Christian-Albrechts-Universität zu Kiel

Signal Processing for Medical Applications –

Frequency Domain Analyses

Muthuraman Muthuraman

Christian-Albrechts-Universität zu Kiel
Department of Neurology / Faculty of Engineering
Digital Signal Processing and System Theory

Dynamic imaging of coherent sources (DICS)

- The power and coherence at any given location in the brain can be computed using a linear transformation which in this case is a spatial filter.
- The spatial filter relates the electromagnetic field on the surface to the underlying neural activity in a certain brain region.
- The neural activity is modeled as a current dipole or sum of current dipoles.

Spatial Filter:

- If we assume x to be a length-N vector containing the potentials measured at the N different electrode sites. The potential due to a single dipole with location vector q is given as:

$$x = H(q) \cdot m(q) \qquad (8.1)$$

where $H(q)$ is an $(N \times 3)-$matrix which is the transfer function and $m(q)$ is a length-3 vector which contains the $(x, y, z)-$components of the dipole moment.

Inverse solution - 1

- If x is due to the potentials of R active dipole sources at locations $q_i,\ i=1,2,\ldots R$ and noise n, then the medium is linear; so the potential at the scalp is the superposition of potentials from the neuronal population. Hence,

$$x=\sum_{i=1}^{R}H(q_i)\cdot m(q_i)+n \tag{8.2}$$

- The spatial filter is designed using the covariance matrix of the data x. Note that x does not contain any temporal information, since it is obtained by sampling the electrodes at a single time instant.

- The x represents the spatial distribution of potential at the measurements sites at the sampling time.

- The electrical activity of a individual neuron is assumed to be random process influenced by external inputs to the neuron.

- Hence, we model the dipole moment as a random quantity and describe its behaviour in terms of mean and covariance.

Inverse solution - 1

- The moment mean vector $\overline{m}(q_i)$ and the covariance $C(q_i)$ are defines as

$$\overline{m}(q_i) = E\{m(q_i)\}, \tag{8.3}$$

$$C(q_i) = E\left\{[m(q_i) - \overline{m}(q_i)][m(q_i) - \overline{m}(q_i)]^T\right\}. \tag{8.4}$$

- If we assume the noise to be zero mean $(E_n = 0)$ with covariance matrix Q, the moments associated to the different dipoles are not correlated, i.e.,

$$E\left\{[m(q_i) - \overline{m}(q_i)][m(q_i) - \overline{m}(q_i)]^T\right\} = 0. \tag{8.5}$$

- Then the mean and the covariance matrix of the observed data vector x can be written as

$$\overline{m}(x) = E(x) = \sum_{i=1}^{L} H(q_i)\, m(q_i) \tag{8.6}$$

- The dipole moment is modeled in terms of mean and covariance $C(q_i)$. The covariance matrix $C(x)$ of measured potentials can be written as:

$$C(x) = E\left\{[x - \overline{m}(x)][x - \overline{m}(x)]^T\right\} = \sum_{i=1}^{R} H(q_i) \cdot C(q_i) \cdot H^T(q_i) + Q \tag{8.7}$$

Inverse solution - 1

- The spatial filter can be defined for a narrow band volume element Q_0 centered at location q_0 as the $(N \times 3)$ matrix $Z(q_0)$ and the three component filter output y can be written as the inner product of $Z(q_0)$ and x:

$$y = Z^T(q_0) \cdot x \tag{8.8}$$

- The linear constraint for an ideal narrowband filter is given as:

$$Z^T(q_0) \cdot H(q) = \begin{cases} I & for\ q \in Q_0 \\ 0 & for\ q \notin Q_0\ \&\ q \in B \end{cases} \tag{8.9}$$

where B represents the volume of the brain. In the absence of noise the filter output is $y = m(q_0)$ where complete attenuation in the stop band is impossible. The optimal filter is $Z(q_0)$ that satisfies these two conditions:

$$\underbrace{\min}_{Z(q_0)} tr\left\{C(y)\right\} \tag{8.10}$$

subject to

$$Z^T(q_0) \cdot H(q) = I \tag{8.11}$$

Inverse solution - 1

- In equation (8.10), the $C(y)$ is given as follows:

$$C(y) = Z^T(q_0) \cdot C(x) \cdot Z(q_0) \tag{8.12}$$

- The solution for the equation (8.11) may be obtained using Lagrange multipliers and completing the square. If $2L_M$ be a (3×3) matrix of Lagrange multipliers.The cost function of the equation (8.12) is augmented with the inner product of the Lagrange multipliers and the constraint to obtain the Lagrangian $L(Z, L_M)$ which is:

$$L(Z, L_M) = tr\left\{Z^T CZ + \left(Z^T H - I\right)2L_M\right\} \tag{8.13}$$

- In this case the arguments q_0 and x are omitted for clarity. Noting that $trB = tr\,B^T$ for any square matrix B, we can rewrite equation (8.13) as

$$L(Z, L_M) = tr\left\{Z^T CZ + \left(Z^T H - I\right)L_M + L_M^T\left(H^T Z - I\right)\right\} \tag{8.14}$$

- It is easy to verify that equation (8.14) can be expressed as the perfect square in Z,

$$L(Z, L_M) = tr\left\{Z^T + \left(L_M^T H^T C^{-1}\right)C\left(Z + C^{-1}HL_M\right) - L_M - L_M^T - L_M^T H^T C^{-1} HL_M\right\} \tag{8.15}$$

Inverse solution - 1

- In the equation (8.15) the first term in the brackets is a fucntion of Z . The matrix C is positive definite; so the minimum of $L(Z, L_M)$ is attained by setting the first term to zero, that is

$$Z = -C^{-1} H L_M \tag{8.16}$$

- The Lagrange multiplier matrix L_M is obtained by substituting Z in the constraint $Z^T H = I$ to obtain

$$L_M^T = -\left(H^T C^{-1} H\right)^{-1} \tag{8.17}$$

Sustituting equation (8.17) into equation (8.16) gives the solution:

$$Z(q_0) = \left(H^T(q_0) \cdot C^{-1}(x) \cdot H(q_0)\right)^{-1} \cdot H^T(q_0) \cdot C^{-1}(x) \tag{8.18}$$

this is the linear constrained minimum variance(LCMV) spatial filter Z as a function of the transfer function H and data covariance matrix C . The main aim of the LMCV method is to design a bank of spatial filters that attenuates signals from other locations and allows only signal generated from a particular location in the brain.

Minimum Norm - 2

- These are the best estimates for the current when minimal a-priori information about the source is available.
- A solution to the difficulty encountered in interpolating or extrapolating magnetic-field or electric-potential maps, the minium norm estimates(MNE) can be used.
- From the original data, the MNE is computed first; the magnetic or electric field at desired points on a given surface can then be calculated directly from the MNE.
- The analysis of the inverse problem from the magnetoencephalography (MEG) data, proposing that a linear combination of magnetometer lead fields should be used as an estimate for the primary-current distribution in the brain.
- The lead field is a vector field that describes the senstivity pattern of a magnetometer to the primary current.

Minimum Norm - 2

- Let us denote the primary current density with J^p:

$$J^p(r) = J_{tot}(r) - \sigma(r)\, E(r) \tag{8.19}$$

where r is the position vector, J_{tot} is the total current density, σ is the conducitivity, and E is the electric field.

- J^p is the result of a change of other types energy into electrical form: it provides the battery of the circuit, driving volume currents (σE) in the conductor.
- We consider σ, J^p and E on a macroscopic scale, so that these quantities are the average or effective values over a volume of about $1\, mm^3$.
- The output of magnetometer B, is linearly related to the primary-current distribution. We can therefore find a vector field $L_i(r)$ satisfying

$$B_i = \int L_i(r) \cdot J^p(r)\, d\upsilon. \tag{8.20}$$

Minimum Norm - 2

- $L_i(r)$ is called the lead field; it describes the sensitivity distribution of the i^{th} magnetometer.

- In addition to the coil configuration of the magnetometer, the lead fields depends on conductivity $\sigma = \sigma(r)$.

- The lead field as defined by equation (8.20), can be computed, provided that it is possible to calculate the magnetic field $B_i = B_i(Q, r')$, resulting from a arbitrary current dipole Q at r'. This requires knowledge of the conductivity distribution $\sigma(r)$, so that effect of volume currents can be properly taken into account.

- For Q at r', $J^p(r) = Q\delta(r - r')$, where $\delta(r)$ is the Dirac delta function.

- Inserting this dipolar primary-current distribution into equation (8.20) we obtain

$$B_i(Q, r') = L_i(r') \cdot Q \tag{8.21}$$

With equation (8.21) all the three components of $L_i(r')$ can be found for any r .

Minimum norm estimate for a current distribution - 2

- In the following, primary-current distributions (in general, continuous) are considered as elements of a function space $\mathscr{F}$ that contains all square-integrable current distributions, confined to a known set of points G inside a conductor; $\mathscr{F}$ is called the current space.

- The set G, in which J^p is confined, which may be a curve, a surface or a volume region.

- When we refer to current distributions as elements of current space, capital letters will be used.

- The inner product of two currents $J_1 \in \mathscr{F}$ and $J_2 \in \mathscr{F}$ is defined by

$$\langle J_1, J_2 \rangle = \int_G J_1(r) \cdot J_2(r)\, dG \qquad (8.22)$$

- The overall amplitude or „length“ of a current distribution is described by ist norm:

$$\|J_k\|^2 = \langle J_k, J_k \rangle = \int_G \left| J_k^p(r) \right|^2 dG \qquad (8.23)$$

Minimum norm estimate for a current distribution - 2

- From equation (8.20), it is evident that measurements $B_i = \langle L_i, J^p \rangle$, $i = 1,\ldots,M$, only yield information about primary currents lying in the subspace $\mathscr{F}$ ' of the current space $\mathscr{F}$.

- This subspace is spanned by the lead fields; $\mathscr{F}$ ' = span $(L_1,\ldots L_M)$. The idea of an MNE is that we search for a estimate J^* for J^p that is confined to $\mathscr{F}$ '. J^* will then be a linear combination of the lead fields:

$$J^* = \sum_{j=1}^{M} w_j L_j \qquad (8.24)$$

where w_j are scalars to be determined from the measurements. Requiring J^* to produce the measured signals $\langle L_i, J^* \rangle = B_i = \langle L_i, J^p \rangle$, we obtain a set of linear equations $b = \Gamma w$, where $b = (B_1,\ldots,B_M)^T, w = (w_1,\ldots,w_M)^T$, and Γ is an $M \times M$ matrix containing the inner products of the lead fields $\Gamma_{ij} = \langle L_i, L_j \rangle$. With this notation equation (8.24) can be compactly written as $J^* = w^T L$ where $L = (L_1,\ldots,L_M)^T$.

- The term minimum-norm estimate derives from the fact that, in the sense of the norm defined by equation (8.23), J^* is the shortest current vector capable of explaining the measured signals.

Minimum norm estimate for a current distribution - 2

- The non-uniqueness of the inverse problem is manifested by the fact that the actual current distribution producing b may be any current of the form $J = J^* + J_{\perp}$, where $J_{\perp}$ satisfies $\langle J_{\perp}, L_i \rangle = 0$, $i = 1, \ldots, M$.

- In other words, any primary-current distribution (together with volume currents) to which the measuring instrument is not senstive may be added to the solution.

Regularisation:

- If the lead fields are linearly independent, which is generally the case when the measurements are made at different locations, the inner product matrix Γ is non -singular and

$$w = \Gamma^{-1} b \tag{8.25}$$

- However, in practice, the $L_i s$ may be nearly linearly dependent. Thus, Γ can possess some very small eigen values, which leads to large errors in the computation of w .

Regularisation

- To avoid this numerical instability, the solution must be regularised. This means that directions in $\mathscr{F}$' with poor coupling to the sensors to be suppressed.

- Let $\Gamma = V \wedge V^T$, with $V^T V = I$ and $\wedge = diag\,(\lambda_1, \ldots \lambda_M)$, where $\lambda_1 > \lambda_2 > \ldots \lambda_M > 0$ are the eigenvalues of Γ. Then $\Gamma^{-1} = V \wedge^{-1} V^T$. Regularisation may be carried out by replacing $\wedge^{-1}$ by $\tilde{\wedge}^{-1} = diag\,(\lambda_1^{-1}, \ldots \lambda_k^{-1}, 0, \ldots, 0)$ to obtain a regularised inverse $\Gamma^{-1} = V \tilde{\wedge}^{-1} V^T$.

- The cut-off value $K \leq M$ is selected so that the regularised MNE does not contain excessive contributions from noise. The MNE does not then exactly reproduce the measured signals, but the misfit $b - \tilde{b}$, where $\tilde{b} = \Gamma \tilde{\omega} = \Gamma \tilde{\Gamma}^{-1} b$, is in accordance with measurement errors.

- In terms of current distributions, regularisation means that those eigenleads that correspond to small eigenvalues, and thus are hard to measure with sufficient signal-to-noise ratio, are ignored. The regularisation minimum norm solution is

$$\tilde{J}^* = \left(\tilde{\Gamma}^{-1} b\right)^T L \qquad (8.26)$$