TECHNISCHE FAKULTÄT DER CHRISTIAN-ALBRECHTS-UNIVERSITÄT ZU KIEL

DIGITALE SIGNALVERARBEITUNG UND SYSTEMTHEORIE

Advanced Signals and Systems

Exam WS 2018

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	25.02.2019
Name:	______________________
Matriculation Number:	______________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorized to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ______________________

Marking

Problem	1	2	3
Points	/37	/33	/30

Total number of points: ____________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated ____________ Signature: ______________________

Advanced Signals and Systems

Exam WS 2018

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Room:	CAP2 - Frederik-Paulsen-Hörsaal
Date:	25.02.2019
Begin:	09:00 h
Reading Time:	10 minutes
Working Time:	90 minutes

Remarks

- Lay out your student or personal ID for inspection.
- Label **each** paper with your **name** and **matriculation number**. Please use a **new sheet of paper** for **each task**. Additional paper is available on request.
- Do **not** use **pencil or red pen**.
- All aids – except for those which allow the communication with another person – are allowed. Prohibited aids are to be kept out of reach and should be turned off.
- The direct communication with any person who is not part of the exam supervision team is prohibited.
- For full credit, your solution is required to be comprehensible and well-reasoned. All sketches of functions require proper labeling of the axes. Please understand that the shown point distribution is only preliminary!
- In case you should feel negatively impacted by your surroundings during the exam, you must notify an exam supervisor immediately.
- The imminent ending of the exam will be announced 5 minutes and 1 minute prior to the scheduled ending time. Once the **end of the exam** has been announced, you **must stop writing** immediately.
- At the end of the exam, put together all solution sheets and hand them to an exam supervisor together with the exam tasks and the **signed cover sheet**.
- Before all exams have been collected, you are prohibited from talking or leaving your seat. Any form of communication at this point in time will still be regarded as an **attempt of deception**.
- During the **reading time, working on the exam tasks is prohibited**. Consequently, all writing tools should remain on your table. Any violation of this rule will be considered as an **attempt of deception**.

Task 1 (37 Points)

Part 1 *This part may be solved independently of parts 2 and 3.*

(a) What is the purpose of signal modulation? Write in one sentence. (2 P)
The modulation is used to adapt the signal spectrum to the frequency response of the transmission medium.

(b) A band limited signal $x(n)$ is multiplied by a sine modulation term $\sin(\Omega_T n)$. Give the spectrum $Y\left(e^{j\Omega}\right)$ of the product of the two signals as an equation. Which modulation type does this implement? (3 P)

$$Y\left(e^{j\Omega}\right) = \frac{1}{2j}\left[X\left(e^{j(\Omega-\Omega_T)}\right) - X\left(e^{j(\Omega+\Omega_T)}\right)\right]$$

with

$$x(n) \circ\!\!-\!\!\bullet X\left(e^{j\Omega}\right).$$

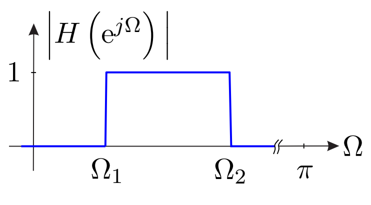
It is a discrete double sideband amplitude modulation without carrier.

(c) What steps must be taken to demodulate a such modulated signal? Describe in full sentences. (3 P)
In the first step, the modulated signal is multiplied by the carrier signal. This shifts the signal spectra back to baseband. The unwanted mixing products are then removed with a low-pass filter.

(d) Which errors can occur during demodulation? (2 P)
Phase errors, frequency errors

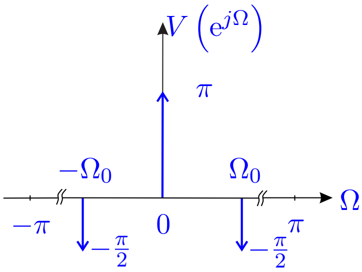
Part 2 *This part may be solved independently of parts 1 and 3.*

The signal $v(n) = \sin^2\left(\frac{\Omega_0}{2}n\right)$ should be transmitted via a channel with a real impulse response and with the following depicted frequency response.



(e) Determine the Fourier transform $V\left(e^{j\Omega}\right) \bullet\!\!-\!\!\circ\, v(n)$ and sketch $V\left(e^{j\Omega}\right)$ in the range of $-\pi$ to π. Assume that $\Omega_0 < \Omega_1$. (5 P)

$$V\left(e^{j\Omega}\right) = \frac{\pi}{2}\left[\sum_{\lambda=-\infty}^{\infty} 2\delta_0(\Omega - 2\pi\lambda) - \delta_0(\Omega + \Omega_0 - 2\pi\lambda) - \delta_0(\Omega - \Omega_0 - 2\pi\lambda)\right].$$



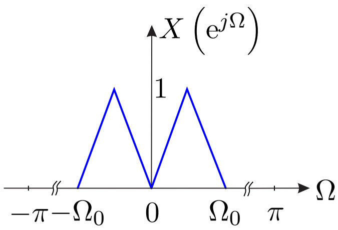
(f) Define the frequency Ω_0 in such a way that the signal $v(n)$ would be transmittable across the channel with respect to just its bandwidth. (2 P)

$$\Omega_0 \leq \frac{\Omega_2 - \Omega_1}{2}$$

(g) Define the range of the carier frequency Ω_T in such a way that the modulated version of $v(n)$ lies in the passband of the channel. Assume that Ω_0 lies in the range as requested in *(f)*. (2 P)

$$\Omega_1 + \Omega_0 \leq \Omega_T \leq \Omega_2 - \Omega_0.$$

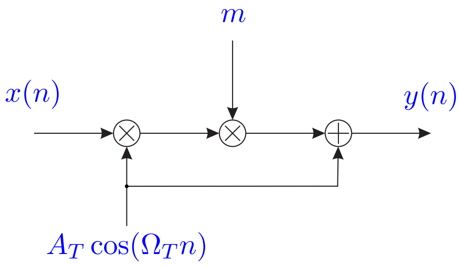
Now the signal $X\left(e^{j\Omega}\right) \bullet\!\!-\!\!\circ\, x(n)$ should be sent via the channel defined above.



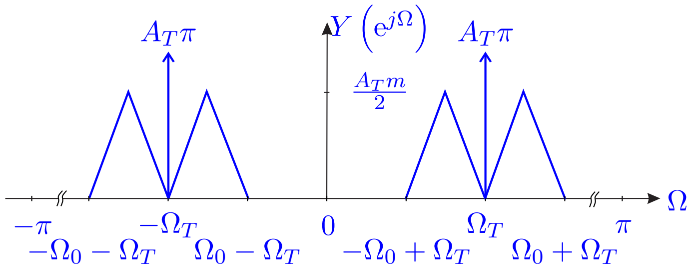
(h) The signal $x(n)$ should be amplitude modulated. Write the general modulation equation. The carrier signal should also be transmitted. (3 P)

$$y(n) = A_T\left[1 + m x(n)\right]\cos(\Omega_T n).$$

(i) Sketch a simple block diagram according to the equation in *(h)*. (4 P)



(j) Sketch the spectrum of the modulated signals from *(h)* in the range of $-\pi$ to π. Assume that $\Omega_0 + \Omega_T < \pi$. (4 P)



Part 3 *This part may be solved independently of parts 1 and 2.*

From the lecture you know the connection between the instantaneous phase and instantaneous frequency. The following equation shows the phase modulation of the carrier $c_{\mathrm{T}}(t)$. $\theta(t)$ stands for the instantaneous phase.

$$c_{\mathrm{T}}(t) = A_{\mathrm{T}} \sin\Big(\theta(t)\Big)$$

Given is the instantaneous frequency $\omega_m(t)$ of the frequency modulated carrier:

$$\omega_m(t) = \begin{cases} \omega_0 + k_{\mathrm{FM}} \sin(2\pi f_0 t), & \text{for } t \geq 0, \\ 0, & \text{else.} \end{cases}$$

(k) Determine the corresponding instantaneous phase $\theta(t)$. (3 P)

$$\theta(t) = \int_0^t \omega_m(\tau) d\tau = \omega_0 t + k_{\mathrm{FM}} \int_0^t \sin(2\pi f_0 \tau) d\tau$$
$$= \omega_0 t + k_{\mathrm{FM}} \Big(1 - \cos(2\pi f_0 t)\Big) \frac{1}{2\pi f_0}.$$

(l) Which amplitude modulation types do you know? Give the advantages and disadvantages of different amplitude modulations. (4 P)

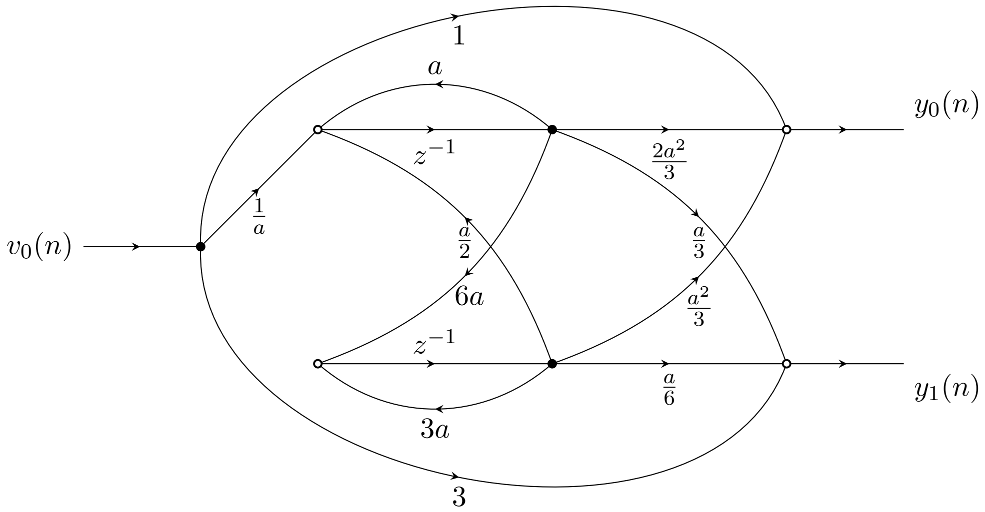
- Double-sideband with and without carrier
- Single-sideband with and without carrier
- generally is AM easy in the implementation

- AM is susceptible to faults
- Double-sideband needs more bandwidth
- Single-sideband needs only half bandwidt

Task 2 (33 Points)

Part 1 *This part may be solved independently of parts 2 and 3.*

For this part a system, which can be described by the signal-flow graph below, is given.



In addition, the following formulae are given:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\,\boldsymbol{x}(n) + \boldsymbol{B}\,\boldsymbol{v}(n), \tag{1}$$

$$\boldsymbol{y}(n) = \boldsymbol{C}\,\boldsymbol{x}(n) + \boldsymbol{D}\,\boldsymbol{v}(n). \tag{2}$$

(a) Determine the number of inputs L, the number of internal states N and the number of outputs R of the system. (1 P)
$L = 1$ inputs, $N = 2$ states, $R = 2$ outputs.

(b) What are equations (1) and (2) denoted by, respectively? What kind of restrictions hold for systems which can be described by these formulae? (2 P)
State and output equations.
Just holds for LTI (linear time-invariant)-systems.

(c) Given equations (1) and (2): (3 P)

(i) Name the matrices (*following the conventions from the lecture*).
$\boldsymbol{A}$ system-matrix, $\boldsymbol{B}$ input-matrix, $\boldsymbol{C}$ output-matrix, $\boldsymbol{D}$ pass-through matrix.

(ii) Determine the dimensions of the matrices according to the signal-flow graph above.

$\boldsymbol{A}$: $[N\text{x}N] = [2\text{x}2]$ $\boldsymbol{B}$: $[N\text{x}L] = [2\text{x}1]$

$\boldsymbol{C}$: $[R\text{x}N] = [2\text{x}2]$ $\boldsymbol{D}$: $[R\text{x}L] = [2\text{x}1]$

(d) Determine the matrices/vectors/scalars $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ for the system given above. (4 P)

$$\boldsymbol{A} = \begin{bmatrix} a & \frac{a}{2} \\ 6a & 3a \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} \frac{1}{a} \\ 0 \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} \frac{2a^2}{3} & \frac{a^2}{3} \\ \frac{a}{3} & \frac{a}{6} \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(e) What must apply to a for the system to be stable? (5 P)

Stability given if poles are within the unit circle: $\|z\| \leq 1$. This statement can be made about the characteristic polynomial. The zeros (i.e. eigenvalues of the matrix $\boldsymbol{A}$) correspond to the poles.

$$N(z) = \det(z\boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} z-a & -\frac{a}{2} \\ -6a & z-3a \end{vmatrix} = (z-a)\ (z-3a) - 3a^2 = z^2 - 4az + 3a^2 - 3a^2$$

It follows $z_{\infty,0,1} = 2a \pm 2a \ \rightarrow \ z_{\infty,0} = 4a, \ z_{\infty,0} = 0$.
With $\|z\| \overset{!}{\leq} 1$ it follows $-\frac{1}{4} \leq a \leq \frac{1}{4}$.

(f) Determine the transfer-matrix of the system. (6 P)
With:

$$\boldsymbol{H}(z) = \boldsymbol{C}\,[z\boldsymbol{I} - \boldsymbol{A}]^{-1}\,\boldsymbol{B} + \boldsymbol{D},$$

and:

$$\boldsymbol{X}^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}^{-1} = \frac{1}{\det(\boldsymbol{X})} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} = \frac{1}{x_{11}\,x_{22} - x_{12}\,x_{21}} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix},$$

you get for the inverse matrix $[z\boldsymbol{I} - \boldsymbol{A}]^{-1}$, where the determinant can be taken from the previous task part:

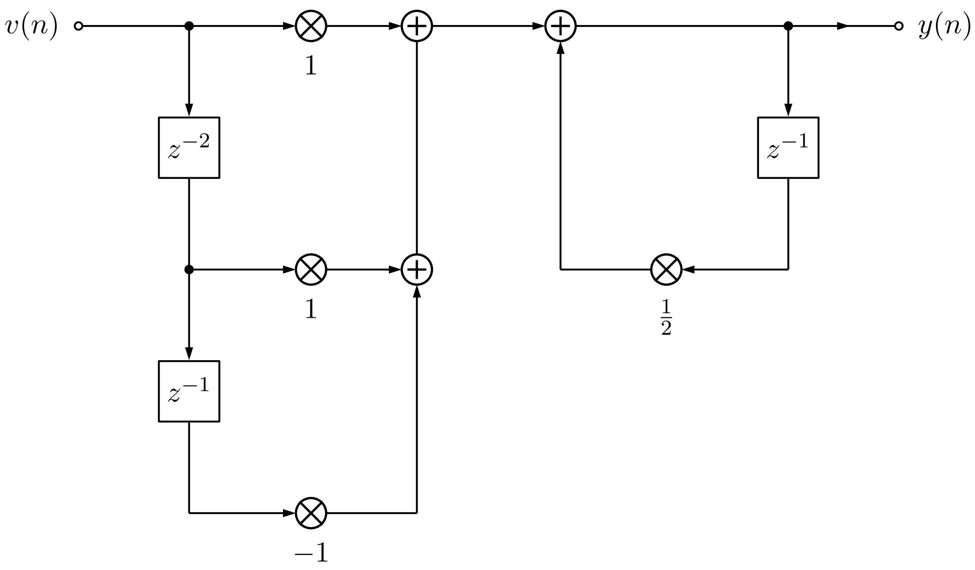
$$[z\boldsymbol{I} - \boldsymbol{A}]^{-1} = \begin{bmatrix} z-a & -\frac{a}{2} \\ -6a & z-3a \end{bmatrix}^{-1} = \frac{1}{z\,(z-4a)} \begin{bmatrix} z-3a & \frac{a}{2} \\ 6a & z-a \end{bmatrix}$$

The transfer matrix is calculated as follows:

$$\begin{aligned} \boldsymbol{H}(z) &= \frac{1}{z\,(z-4a)} \begin{bmatrix} \frac{2a^2}{3} & \frac{a^2}{3} \\ \frac{a}{3} & \frac{a}{6} \end{bmatrix} \begin{bmatrix} z-3a & \frac{a}{2} \\ 6a & z-a \end{bmatrix} \begin{bmatrix} \frac{1}{a} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \frac{1}{z\,(z-4a)} \begin{bmatrix} \frac{2a}{3}\,z \\ \frac{1}{3}\,z \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3}\,\frac{a}{z-4a} + 1 \\ \frac{1}{3}\,\frac{1}{z-4a} + 3 \end{bmatrix} \end{aligned}$$

Part 2 *This part may be solved independently of parts 1 and 3.*

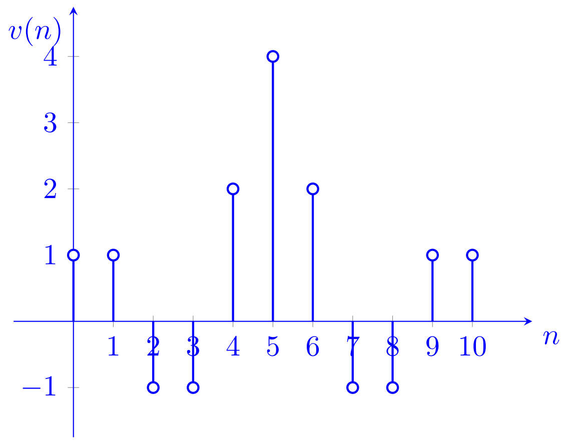
For this part of the task, a system is given, which is described by following block diagram. It is also true that all memories for $n < 0$ are initialized with 0 and $y(n) = 0 \ \forall n < 0$.



First a signal $v(n)$ is defined as follows:

$$v(n) = \gamma_{-1}(n) - 2\,\gamma_{-1}(n-2) + 3\,\gamma_{-1}(n-4) + 2\,\gamma_0(n-5) - 3\,\gamma_{-1}(n-7) + 2\left[\gamma_0(n-9) + \gamma_0(n-10)\right] + \gamma_{-1}(n-11).$$

(g) Draw $v(n)$ for the exact range $0 \le n < 11$. (2 P)



Now the system is excited with the following signal:

$$v(n) = \gamma_{-1}(n) - \gamma_{-1}(n-1).$$

(h) Determine the output signal $y(n)\ \forall n \in \mathbb{N}$. (4 P)
Due to feedback, the use of numerical values does not make sense. Reading from the block diagram shows:

$$y(n) = \frac{1}{2}y(n-1) + v(n) + v(n-2) - v(n-3).$$

Kronecker-Delta as input signal:

$$v(n) = \gamma_{-1}(n) - \gamma_{-1}(n-1) = \gamma_0(n).$$

Differential equation:

$$y(n) - \frac{1}{2}y(n-1) = v(n) + v(n-2) - v(n-3).$$

z-transform and $H(z) = Y(z)/V(z)$ provide:

$$H(z) = \frac{1 + z^{-2} - z^{-3}}{1 - \frac{1}{2}z^{-1}} = \frac{z + z^{-1} - z^{-2}}{z - \frac{1}{2}} = \frac{z}{z - \frac{1}{2}} + \frac{z^{-1}}{z - \frac{1}{2}} - \frac{z^{-2}}{z - \frac{1}{2}}.$$

Inverse transform with known correspondences gives:

$$h_0(n) = \left(\frac{1}{2}\right)^n \gamma_{-1}(n) + \left(\frac{1}{2}\right)^{n-2} \gamma_{-1}(n-2) - \left(\frac{1}{2}\right)^{n-3} \gamma_{-1}(n-3).$$

As the system is excited with an impuls/Kronecker delta, $y(n)$ describes the impulse response $y(n) = h_0(n)$.

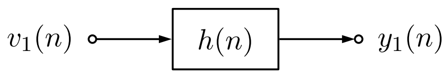
(i) How is this special signal called? Justify your answer. (1 P)
Impulse response, as system is excited with $\gamma_0(n)$.

Part 3 *This part may be solved independently of parts 1 and 2.*

Now the output signal

$$y_1(n) = \gamma_0(n) + a^n\ \gamma_{-1}(n-2) + b\ \gamma_{-1}(n-4)$$

is given. Furthermore, $v_1(n) = \gamma_{-1}(n)$ and $a, b \in \mathbb{R}$ with $a < 1$ hold.



(j) Determine $H(z)|_{z=1}$. (4 P)
As the input signal of the system is the step sequence the corresponding output signal describes the step response of the system.

$$y_1(n) = h_{-1}(n).$$

Due to this the following limit theorem can be utilized to determine $H(z)|_{z=1}$:

$$H(z)|_{z=1} = \lim_{n\to\infty} h_{-1}(n) = b,$$

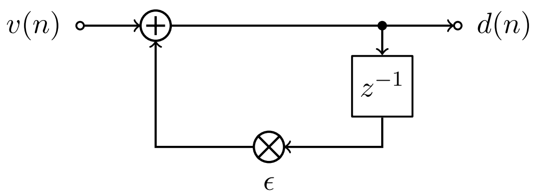
as $\lim_{n\to\infty} \gamma_{-1}(n) = 1$, $\lim_{n\to\infty} a = 0$ for $a < 1$.

(k) What does $H(z)|_{z=1}$ describe? (1 P)
It describes the direct component of a system, as $z = \rho\ \exp\{j\Omega\} \underset{\Omega=0}{=} 1$.

Task 3 (30 Points)

Part 1 *This part may be solved independently of part 2.*

Given is the following system with real impulse response $h_0(n)$ and a real constant $|\epsilon| < 1$:



In the following, the system is excited with zero-mean, white noise of power σ_v^2.
Hint: Use the statistical definitions of the queried values.

(a) Give the autocorrelation function of process $v(n)$. (1 P)

$$s_{vv}(\kappa) = \sigma_v^2 \cdot \gamma_0(\kappa)$$

(b) Compute the power of process $d(n)$ in dependence of σ_v^2. (3 P)
It holds:

$$\begin{aligned} m_d^{(2)} &= E\Big\{d^2(n)\Big\} = E\Big\{[v(n) + \epsilon d(n-1)]^2\Big\} \\ &= E\Big\{v^2(n)\Big\} + \epsilon \cdot E\Big\{v(n)d(n-1)\Big\} + \epsilon^2 \cdot E\Big\{d^2(n-1)\Big\} \\ &= \sigma_v^2 + 0 + \epsilon^2 m_d^{(2)} \end{aligned}$$

From this follows $m_d^{(2)} = \frac{\sigma_v^2}{1-\epsilon^2}$.

(c) Compute the autocorrelation function $s_{dd}(\kappa)$ of $d(n)$ at $\kappa = 1, 2$. (4 P)

$$s_{dd}(1) = E\Big\{d(n)d(n-1)\Big\} = E\Big\{[v(n) + \epsilon d(n-1)]d(n-1)\Big\} = \epsilon m_d^{(2)}$$

$$s_{dd}(2) = E\Big\{d(n)d(n-2)\Big\} = E\Big\{[v(n) + \epsilon[v(n-1) + \epsilon d(n-2)]]d(n-2)\Big\} = \epsilon^2 m_d^{(2)}$$

(d) Give the function $s_{dd}(\kappa)$ for all κ by using your results from *(b,c)*. (2 P)

$$s_{dd}(\kappa) = m_d^{(2)} \cdot \epsilon^{|\kappa|} = \frac{\sigma_v^2}{1-\epsilon^2} \cdot \epsilon^{|\kappa|}$$

(e) What effect does the system have on $v(n)$? What exactly is influenced by ϵ? (2 P)
The system turns white noise into colored noise; hence, the system introduces correlation into the signal. The parameter ϵ steers the degree of correlation to be expected from the colored noise.

(f) Determine $h_0(n) * h_0(-n)$. (3 P)
It holds: $s_{dd}(\kappa) = s_{vv}(\kappa) * h_0(\kappa) * h_0(-\kappa) = \sigma_v^2 \cdot h_0(\kappa) * h_0(-\kappa)$.
From this follows $h_0(n) * h_0(-n) = \frac{\epsilon^{|n|}}{1-\epsilon^2}$.

Part 2 *This part may be solved independently of part 1.*

Given is the following joint density function of the real stochastic variables x and y with the real constants α and β:

$$f_{xy}(x,y) = \delta_0\Big(x^2+y^2-1\Big)\cdot\sin\Big(\varphi(x)\Big)\cdot\begin{cases}\alpha, & \text{for } y\geq 0,\\ \beta, & \text{for } y<0.\end{cases}$$

Due to $\delta_0(...)$, the density is different from zero solely on the unit circle of the x-y-plane. For the set of unit circle coordinate pairs the argument $\varphi(x)$ is defined by the x-coordinate and the inverse of the cosine:

$$\varphi(x) = \arccos(x)$$

Consequently, the constraint $\varphi\in[0,\pi]$ holds.

(g) Which two constraints must hold for the range of the constants α und β? Give reason to your answer! (2 P)
$\alpha,\beta\geq 0$ since densities cannot be negative at any point; at least one constant must be greater than zero, so the integral over the entire density may equal 1.

(h) Compute β given $\alpha=\frac{1}{6}$. (4 P)
Generally,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)dxdy \overset{!}{=} 1$$

holds. By performing a coordinate transformation, the integral may be written as

$$\alpha\int_0^{\pi}\int_0^{\infty}\delta_0(r^2-1)\sin(\varphi)rdrd\varphi + \beta\int_{\pi}^{2\pi}\int_0^{\infty}\delta_0(r^2-1)\sin(2\pi-\varphi)rdrd\varphi.$$

Due to the sifting property of the Dirac impulse, the dependence on r may be eliminated:

$$\alpha\int_0^{\pi}\sin(\varphi)d\varphi + \beta\int_{\pi}^{2\pi}\sin(2\pi-\varphi)d\varphi.$$

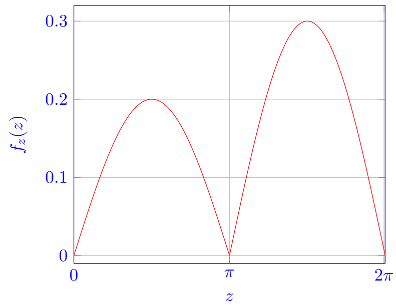
Hence, it follows:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)dxdy &= (\alpha+\beta)\int_0^{\pi}\sin(\varphi)d\varphi\\ &= (\alpha+\beta)\Big[-\cos(\varphi)\Big]_0^{\pi}\\ &= 2(\alpha+\beta).\end{aligned}$$

This leads to $\beta=\frac{1}{3}$ by applying above approach. The solution may also gain full credit without the formal coordinate transform, since the integral obviously breaks down into $(\alpha+\beta)\int_0^{\pi}\sin(\varphi)d\varphi$ due to the information given in the task.

Now, a new stochastic variable $z\in[0,2\pi)$ defined as the phase of the complex number $c=x+jy$ is given. In the follwong, please use $\alpha=\frac{1}{5}$ and $\beta=\frac{3}{10}$.

(i) Sketch the corresponding density $f_z(z)$. (3 P)



(j) Compute the first statistical moment of z. (4 P)
Hint: $\int g \sin(g)\, dg = \sin(g) - g\cos(g) + C$.

$$\begin{aligned} E\{z\} &= \int_0^{2\pi} z f_z(z) dz = \int_0^{\pi} z \frac{1}{5} \sin(z) dz - \int_{\pi}^{2\pi} z \frac{3}{10} \sin(z) dz \\ &= \frac{1}{5}\Big[\sin(z) - z\cos(z)\Big]_0^{\pi} - \frac{3}{10}\Big[\sin(z) - z\cos(z)\Big]_{\pi}^{2\pi} \\ &= \frac{1}{5}[0 + \pi - 0 - 0] - \frac{3}{10}[0 - 2\pi - 0 - \pi] = \frac{11}{10}\pi \end{aligned}$$

Additionally, the stochastic process $\gamma = \sin(z + 3\pi \cdot t)$ with continuous time variable $t \in \mathbb{R}$ is given.

(k) Is the process γ ergodic? Give reason to your answer! (2 P)
Since z is not equally distributed, the ensemble average is dependent on time and is (except for certain periodic time instances) different from zero. The time average, however, is always zero and independent of the process instance. Consequently, the process is not ergodic.

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