Advanced Signals and Systems

Exam WS 2017/2018

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	16.03.2018
Name:	________________________________
Matriculation Number:	________________________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorized to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ________________________________

Marking

Problem	1	2	3
Points	/34	/33	/33

Total number of points: ________________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated ________________ Signature: ________________________________

Advanced Signals and Systems

Exam WS 2017/2018

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	16.03.2018
Time:	09:00 h – 10:30 h (90 minutes)
Location:	KS2/Geb.C - SR II

Remarks

- Please write your **name** and your **matriculation number** on each sheet of paper that you return.
- Please keep your student ID and your identity card ready.
- During the exam only questions concerning the problems are answered.
- Please don't use any pencil or red pen.
- Please use a **new** sheet of paper with your name and matriculation number on it for **each problem**. You can ask for more sheets of paper, if necessary.
- The exam is open books, open notes; other people are closed. Programmable electronical devices except pocket calculators are not permitted.
- Partial credit will be given. No credit will be given if an answer appears with no supporting work or reason.
- Note that the given points of the subproblems are just preliminary.
- At the end of the exam put all sheets together as you have received them, including the problem sheets.
- No one is allowed to talk or to leave his or her seat until **all** exams have been collected.
- The problems and the solutions will be published on the website of the lecture. Also the date and the place of the inspection will be announced on this website.

Problem 1 (34 points)

Part 1 *This part may be solved independently of parts 2 and 3.*

Given is the probability function of a random process with unknowns $a, b, c \in \mathbb{R}$ and Euler's number e:

$$F_x(x) = \begin{cases} a, & \text{for } x < 1 \\ \ln(5b \cdot x), & \text{for } 1 \leq x \leq e \\ c, & \text{else} \end{cases}$$

(a) Determine the unknowns a, b, c. (3 P)
$a = 0,\ b = \frac{1}{5},\ c = 1$

(b) Calculate the corresponding probability density function using the results from *(a)*. (3 P)
It holds: $f_x(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_x(x)$.

$$f_x(x) = \begin{cases} 0, & \text{for } x < 1 \\ \frac{1}{x}, & \text{for } 1 \leq x \leq e \\ 0, & \text{else} \end{cases}$$

(c) Calculate both the first and second statistical moment of the random process. (6 P)

$$\begin{aligned} \mathrm{E}\{x\} &= \int_{-\infty}^{\infty} x \cdot f_x(x)\,\mathrm{d}x = \int_1^e \frac{x}{x}\,\mathrm{d}x \\ &= \Big[x\Big]_1^e = e - 1 \end{aligned}$$

$$\begin{aligned} \mathrm{E}\{x^2\} &= \int_{-\infty}^{\infty} x^2 \cdot f_x(x)\,\mathrm{d}x = \int_1^e \frac{x^2}{x}\,\mathrm{d}x \\ &= \left[\frac{x^2}{2}\right]_1^e = \frac{e^2 - 1}{2} \end{aligned}$$

(d) Calculate the second central statistical moment of the random process. (2 P)

$$\begin{aligned} \mathrm{E}\left\{(x - \mathrm{E}\{x\})^2\right\} &= \mathrm{E}\{x^2\} - \mathrm{E}^2\{x\} \\ &= \frac{e^2 - 1}{2} - (e - 1)^2 \\ &= 2e - \frac{e^2}{2} - 1.5 \end{aligned}$$

Part 2 *This part may be solved independently of parts 1 and 3.*

Given is the following joint probability density function:

$$f_{v_1,v_2}(v_1,v_2) = \begin{cases} \alpha \cdot v_1 \cdot e^{-\beta\left(v_2-\gamma\right)^2}, & \text{for } 2 \le v_1 \le 3 \ \wedge \ v_2 \in \mathbb{R} \\ 0, & \text{else} \end{cases}$$

The degrees of freedom α, β, γ are, at first, unknown and are assumed to be real numbers.

(e) Calculate the marginal probability density function $f_{v_2}(v_2)$. (3 P)

$$\begin{aligned} f_{v_2}(v_2) &= \int_2^3 f_{v_1,v_2}(v_1,v_2)\,\mathrm{d}v_1 \\ &= \left[\alpha \cdot \frac{v_1^2}{2} \cdot e^{-\beta\left(v_2-\gamma\right)^2}\right]_{v_1=2}^{v_1=3} \\ &= \frac{5\,\alpha}{2} \cdot e^{-\beta\left(v_2-\gamma\right)^2}, \ v_2 \in \mathbb{R} \end{aligned}$$

(f) Determine α in such a way that $f_{v_2}(v_2)$ resembles a normal distribution. (3 P)

Comparing with the known formula of the normal distribution yields $\frac{5\,\alpha}{2} \stackrel{!}{=} \frac{1}{\sqrt{2\pi\sigma^2}}$ and $\beta \stackrel{!}{=} \frac{1}{2\sigma^2}$. From this follows $\alpha = \frac{2}{5} \cdot \sqrt{\frac{\beta}{\pi}}$.

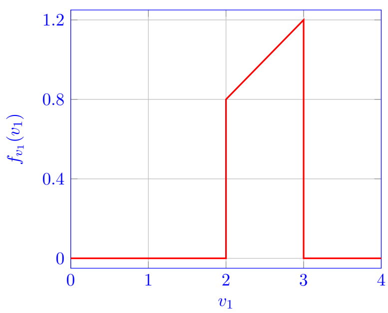
(g) Determine $f_{v_1}(v_1)$ and sketch your result in an appropriate interval. You may assume that v_1 und v_2 are statistically independent. (4 P)

It holds: $f_{v_1,v_2}(v_1,v_2) = f_{v_1}(v_1) \cdot f_{v_2}(v_2)$.

$$f_{v_1,v_2}(v_1,v_2) = \begin{cases} \frac{2}{5}v_1 \cdot \frac{5\,\alpha}{2} \cdot e^{-\beta\left(v_2-\gamma\right)^2} \\ 0 \end{cases} = \begin{cases} \frac{2}{5}v_1 \cdot f_{v_2}(v_2), & \text{for } 2 \le v_1 \le 3 \ \wedge \ v_2 \in \mathbb{R} \\ 0, & \text{else} \end{cases}$$

This yields:

$$f_{v_1}(v_1) = \begin{cases} \frac{2}{5}v_1, & \text{for } 2 \le v_1 \le 3 \\ 0, & \text{else} \end{cases}$$



Part 3 *This part may be solved independently of parts 1 and 2.*

The real output signal $y(n)$ of an LTI system may be described by the following equation, where $v(n)$ is the real input signal of the system:

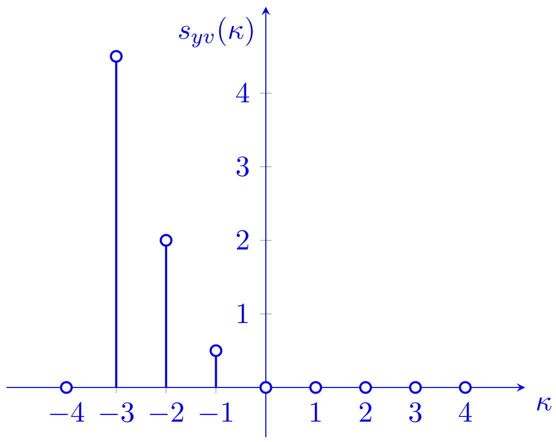
$$y(n) = \sum_{i=-\infty}^{\infty} h_0(i)\, v(n-i), \quad h_0(i) \in \mathbb{R}.$$

(h) Give an equation for the relationship of the correlation functions $s_{yv}(\kappa)$ and $s_{vv}(\kappa)$ of the signals, assuming stationary excitation. How may this equation be rewritten using a convolution operator? (2 P)

$$s_{yv}(\kappa) = \sum_{i=-\infty}^{\infty} h_0(-i)\, s_{vv}(\kappa - i) = h_0(-\kappa) * s_{vv}(\kappa)$$

Now, such a system is excited with ideal white noise ($m_v = 0,\ \sigma_v^2 = 0.5$). Additionally, $h_0(i) = i^2$ holds in the interval $0 < i < 4$ and $h_0(i) = 0$ holds everywhere else.

(i) Sketch $s_{yv}(\kappa)$. Supply all necessary information! (5 P)



(j) Calculate $S_{yv}(e^{j\Omega})$. (3 P)
Use correspondences:

$$S_{yv}(e^{j\Omega}) = \mathscr{F}\{s_{yv}(\kappa)\} = 0.5e^{j\Omega} + 2e^{j2\Omega} + 4.5e^{j3\Omega}$$

Problem 2 (33 points)

Part 1 *This part may be solved independently of parts 2 and 3.*

Given is a difference equation of a linear time-invariant system with input $v(n)$ and output $y(n)$:

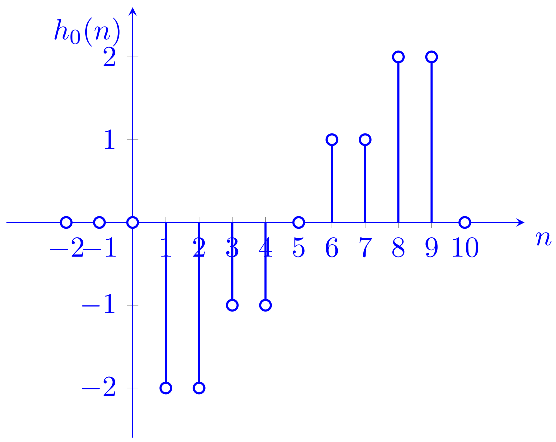
$$y(n) = y(n-1) - 2v(n-1) + v(n-3) + v(n-5) + v(n-6) + v(n-8) - 2v(n-10).$$

(a) Determine the transfer function $H(z)$ of the system. (3 P)

$$H(z) = \frac{Y(z)}{V(z)} = \frac{-2z^{-1} + z^{-3} + z^{-5} + z^{-6} + z^{-8} - 2z^{-10}}{1 - z^{-1}}$$
$$= -2z^{-1}\frac{z}{z-1} + z^{-3}\frac{z}{z-1} + z^{-5}\frac{z}{z-1} + z^{-6}\frac{z}{z-1} + z^{-8}\frac{z}{z-1} - 2z^{-10}\frac{z}{z-1}$$

(b) Determine the impulse response of the system. Sketch it for $-3 < n < 11$. (5 P)
Inverse z-Transformation of result from *(a)* (or excitation with $v(n) = \gamma_0$) yields:

$$h_0(n) = -2\gamma_{-1}(n-1) + \gamma_{-1}(n-3) + \gamma_{-1}(n-5) + \gamma_{-1}(n-6) + \gamma_{-1}(n-8) - 2\gamma_{-1}(n-10)$$



(c) Does the system have a direct pass-through? Give reason to your answer using your result from *(b)*. (1 P)
No, because $h(0) = 0$.

Part 2 *This part may be solved independently of parts 1 and 3.*

Now the following equation of a linear time-invariant system with input $v(n)$ and output $y(n)$ is given:

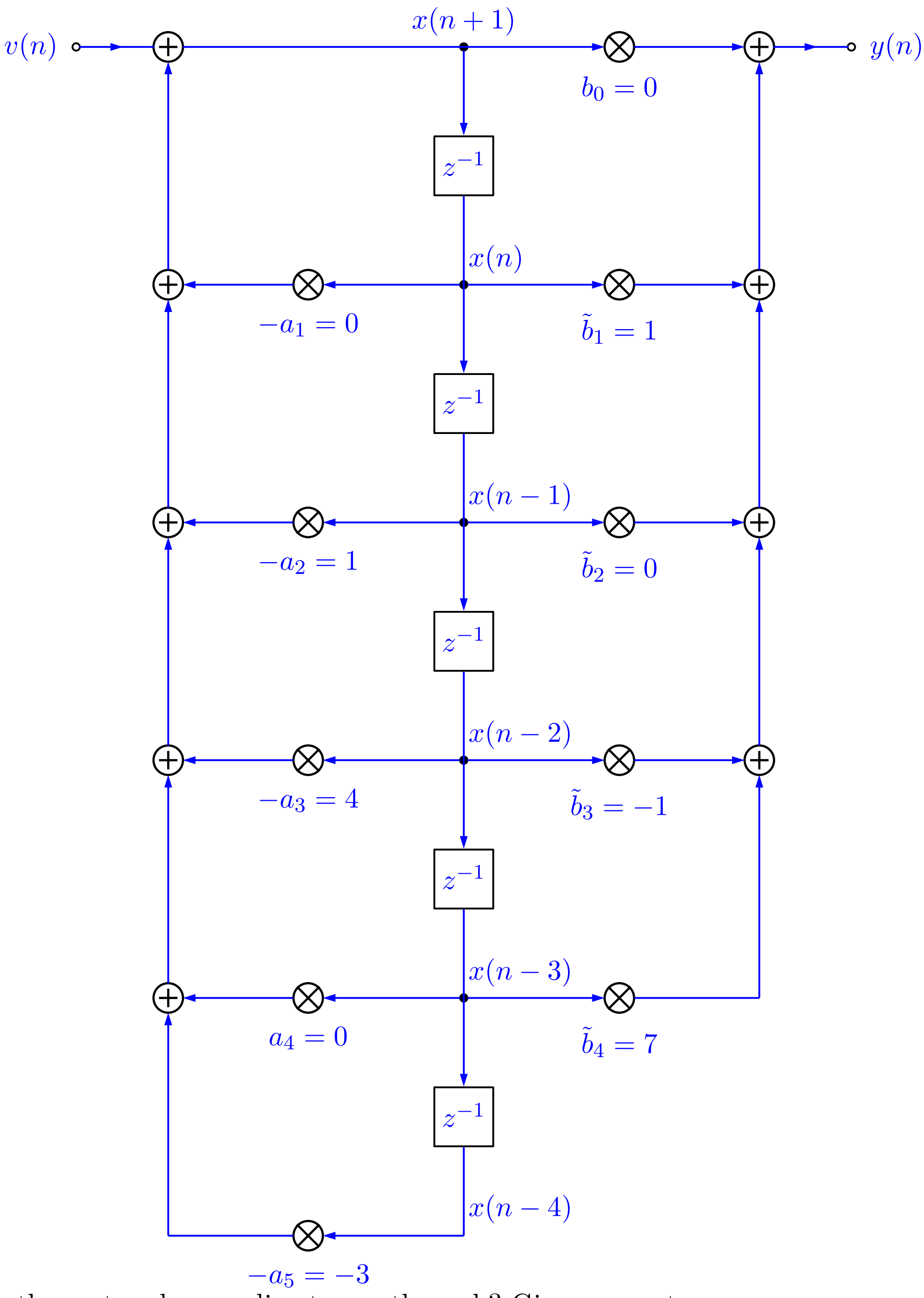
$$y(n+4) = y(n+2) + 4y(n+1) - 3y(n-1) + v(n+3) - v(n+1) + 7v(n)$$

(d) Draw the signalflow graph of Direct Form II. (5 P)

Because of LTI-System the equation can be rewritten:

$$y(n) = y(n-2) + 4y(n-3) - 3y(n-5) + v(n-1) - v(n-3) + 7v(n-4)$$

Direct Form II:



(e) Does the system have a direct pass-through? Give reason to your answer. (1 P)
No, because $v(n)$ has no direct influence on $y(n)$ ($b_0 = 0$).

(f) Determine the state-space matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ with respect to the definition of the lecture. Give your definition of the state-space vector with respect to task *(d)*. (4 P)

Due to script (VII-20) and the above definition (part *(d)*) for the states:

$$\mathbf{x} = \begin{bmatrix} x(n) & x(n-1) & x(n-2) & x(n-3) & x(n-4) \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 & -a_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & 0 & -3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

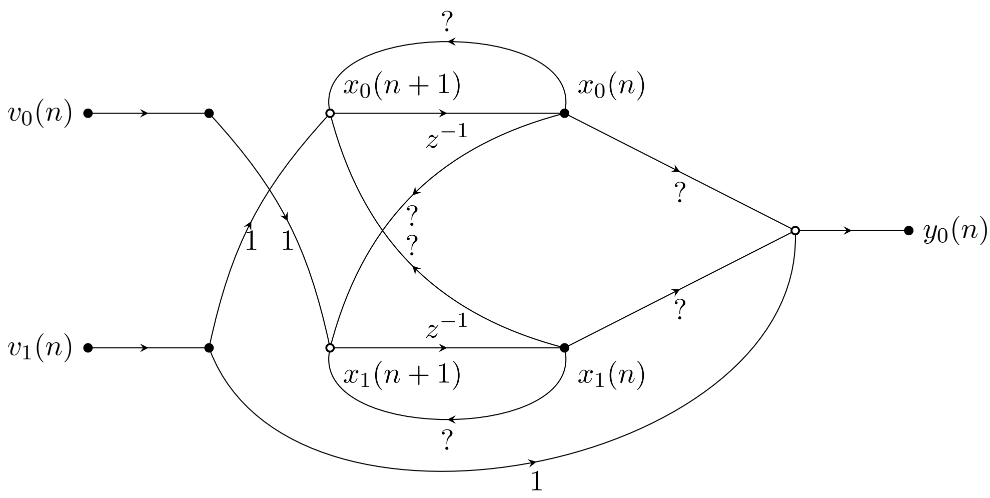
$$\mathbf{c} = \begin{bmatrix} \tilde{b}_1 - b_0 a_1 & \tilde{b}_2 - b_0 a_2 & \tilde{b}_3 - b_0 a_3 & \tilde{b}_4 - b_0 a_4 & \tilde{b}_5 - b_0 a_5 \end{bmatrix}^{\mathrm{T}}$$

$$= \begin{bmatrix} 1 & 0 & -1 & 7 & 0 \end{bmatrix}^{\mathrm{T}}$$

$$d = b_0 = 0$$

Part 3 *This part may be solved independently of parts 1 and 2.*

Given is the signalflow graph:



Additionally, the states $\mathbf{x}(n) = [x_0(n), x_1(n)]^{\mathrm{T}}$ and the output signal $y_0(n)$ for $n < 4$ are known:

$$x_0(n) = \begin{cases} 0 & , n \leq 1, \\ 1,5 & , n = 2, \\ 2 & , n = 3, \\ \dots \end{cases} \qquad x_1(n) = \begin{cases} 0 & , n < 1, \\ 1 & , n = 1, \\ 1 & , n = 2, \\ 1,75 & , n = 3, \\ \dots \end{cases}$$

$$y_0(n) = \begin{cases} 0 & , n < 1, \\ 4 & , n = 1, \\ 4,5 & , n = 2, \\ 7,25 & , n = 3, \\ \dots \end{cases}$$

Furthermore, $\mathbf{v}(n) = [v_0(n), v_1(n)]^{\mathrm{T}} = [\gamma_0(n), \gamma_0(n-1)]^{\mathrm{T}}$ is given, where $\gamma_0(n)$ is the Kronecker-Delta.

(g) Give the general state-space equations from the lecture. (1 P)
In general the state-space equations are given as:

$$\mathbf{x}(n+1) = \mathbf{A}\,\mathbf{x}(n) + \mathbf{B}\,\mathbf{v}(n)$$
$$\mathbf{y}(n) = \mathbf{C}\,\mathbf{x}(n) + \mathbf{D}\,\mathbf{v}(n)$$

(h) Name one popular application for the state-space description of a system. (1 P)
E.g. Kalman-Filter (e.g. for tracking,...).

(i) Which restrictions hold for systems which could be described by fixed parameters in the state-space domain? (1 P)
LTI System → linear, time-invariant system.

(j) Determine the matrices **A**, **B**, **C**, **D** of the state-space description and name them. (11 P)
The signalflow graph directly yields:

$$\mathbf{A} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$\mathbf{C} = \mathbf{c} = \begin{bmatrix} ? & ? \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \end{bmatrix},$$
$$\mathbf{D} = \mathbf{d} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

- Calculation of entries for matrix **C** using the state-space description from part *(g)*:

$$y_0(n) = c_1 x_0(n) + c_2 x_1(n) + v_1(n)$$

  Inserting values for $n = 1$ yields $c_2 = 3$. Using this result and the values for $n = 2$ yields $c_1 = 1$. And such:

$$\mathbf{c} = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

- Calculation of the entries of matrix **A** using the state-space description of *(g)*:

$$x_0(n+1) = a_{11} x_0(n) + a_{12} x_1(n) + v_2(n)$$
$$x_1(n+1) = a_{21} x_0(n) + a_{22} x_1(n) + v_1(n)$$

  Inserting values for $n = 1$ yields $a_{12} = 0.5$ and $a_{22} = 1$. Using a_{12} and a_{22} and inserting values for $n = 2$ yields $a_{11} = 1$ and $a_{21} = 0.5$ and such:

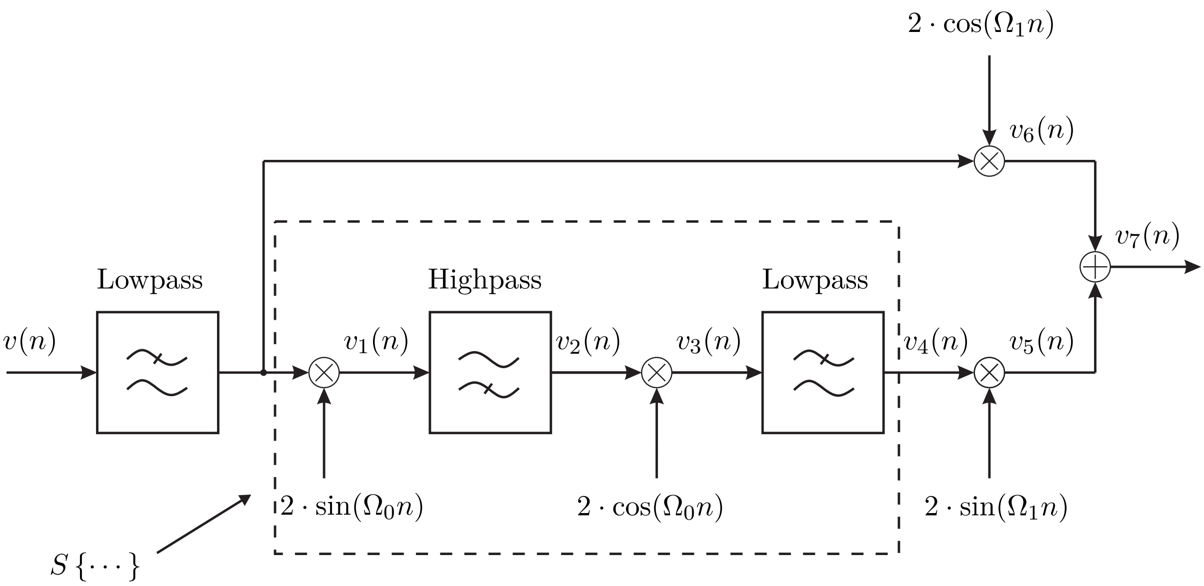
$$\mathbf{A} = \begin{bmatrix} 1 & 0,5 \\ 0,5 & 1 \end{bmatrix}$$

A system matrix, **B** input matrix, **C** output matrix, **D** pass-through matrix.

Problem 3 (33 points)

Part 1 *This part may be solved independently of part 2.*

Given is a discrete modulation system according to the following block diagram:



where $v(n)$ is the signal to be modulated. The cut-off frequencies of the ideal high and low passes lie identically at Ω_0.

(a) Determine the signals $v_1(n)$ to $v_7(n)$ (7 signals) for the case that the input signal $v(n)$ is a cosine: (7 P)

$$v(n) = \cos(\Omega_m n)$$

with $0 < \Omega_m < \Omega_0$ and $\Omega_1 > \Omega_m$.

$$\begin{aligned}
v(n) &= \cos(\Omega_m n), \\
v_1(n) &= v(n) \cdot 2 \cdot \sin(\Omega_0 n) = \cos(\Omega_m n) \cdot 2 \cdot \sin(\Omega_0 n) \\
&= \sin\Big((\Omega_0 - \Omega_m)n\Big) + \sin\Big((\Omega_0 + \Omega_m)n\Big), \\
v_2(n) &= \sin\Big((\Omega_0 + \Omega_m)n\Big), \\
v_3(n) &= \sin\Big((\Omega_0 + \Omega_m)n\Big) \cdot 2 \cdot \cos(\Omega_0 n) \\
&= \sin(\Omega_m n) + \sin\Big((2 \cdot \Omega_0 + \Omega_m)n\Big), \\
v_4(n) &= \sin(\Omega_m n), \\
v_5(n) &= \sin(\Omega_m n) \cdot 2 \cdot \sin(\Omega_1 n) \\
&= \cos\Big((\Omega_1 - \Omega_m)n\Big) - \cos\Big((\Omega_1 + \Omega_m)n\Big), \\
v_6(n) &= \cos(\Omega_m n) \cdot 2 \cdot \cos(\Omega_1 n) \\
&= \cos\Big((\Omega_1 - \Omega_m)n\Big) + \cos\Big((\Omega_1 + \Omega_m)n\Big), \\
v_7(n) &= v_6(n) + v_5(n) \\
&= 2 \cdot \cos\Big((\Omega_1 - \Omega_m)n\Big).
\end{aligned}$$

(b) Determine the Fourier transforms $V_1\left(e^{j\Omega}\right)$ to $V_7\left(e^{j\Omega}\right)$ (7 spectra). (7 P)

$$V\left(e^{j\Omega}\right) = \pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + \Omega_m - 2\pi k) + \gamma_0(\Omega - \Omega_m - 2\pi k)],$$

$$V_1\left(e^{j\Omega}\right) = j\pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + (\Omega_0 - \Omega_m) - 2\pi k) - \gamma_0(\Omega - (\Omega_0 - \Omega_m) - 2\pi k)] + j\pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + (\Omega_0 + \Omega_m) - 2\pi k) - \gamma_0(\Omega - (\Omega_0 + \Omega_m) - 2\pi k)],$$

$$V_2\left(e^{j\Omega}\right) = j\pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + (\Omega_0 + \Omega_m) - 2\pi k) - \gamma_0(\Omega - (\Omega_0 + \Omega_m) - 2\pi k)],$$

$$V_3\left(e^{j\Omega}\right) = j\pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + \Omega_m - 2\pi k) - \gamma_0(\Omega - \Omega_m - 2\pi k)] + j\pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + (2\cdot\Omega_0 + \Omega_m) - 2\pi k) - \gamma_0(\Omega - (2\cdot\Omega_0 + \Omega_m) - 2\pi k)],$$

$$V_4\left(e^{j\Omega}\right) = j\pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + \Omega_m - 2\pi k) - \gamma_0(\Omega - \Omega_m - 2\pi k)],$$

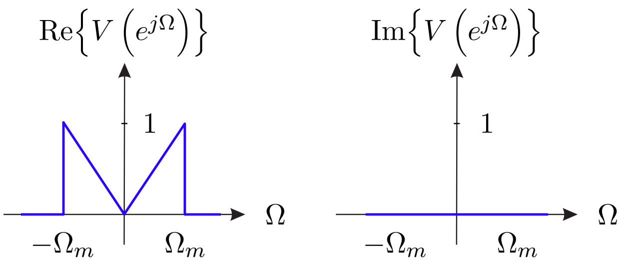
$$V_5\left(e^{j\Omega}\right) = \pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + (\Omega_1 - \Omega_m) - 2\pi k) + \gamma_0(\Omega - (\Omega_1 - \Omega_m) - 2\pi k)] - \pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + (\Omega_1 + \Omega_m) - 2\pi k) + \gamma_0(\Omega - (\Omega_1 + \Omega_m) - 2\pi k)],$$

$$V_6\left(e^{j\Omega}\right) = \pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + (\Omega_1 - \Omega_m) - 2\pi k) + \gamma_0(\Omega - (\Omega_1 - \Omega_m) - 2\pi k)] + \pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + (\Omega_1 + \Omega_m) - 2\pi k) + \gamma_0(\Omega - (\Omega_1 + \Omega_m) - 2\pi k)],$$

$$V_7\left(e^{j\Omega}\right) = 2\cdot\pi \sum_{k=-\infty}^{\infty} [\gamma_0(\Omega + (\Omega_1 - \Omega_m) - 2\pi k) + \gamma_0(\Omega - (\Omega_1 - \Omega_m) - 2\pi k)].$$

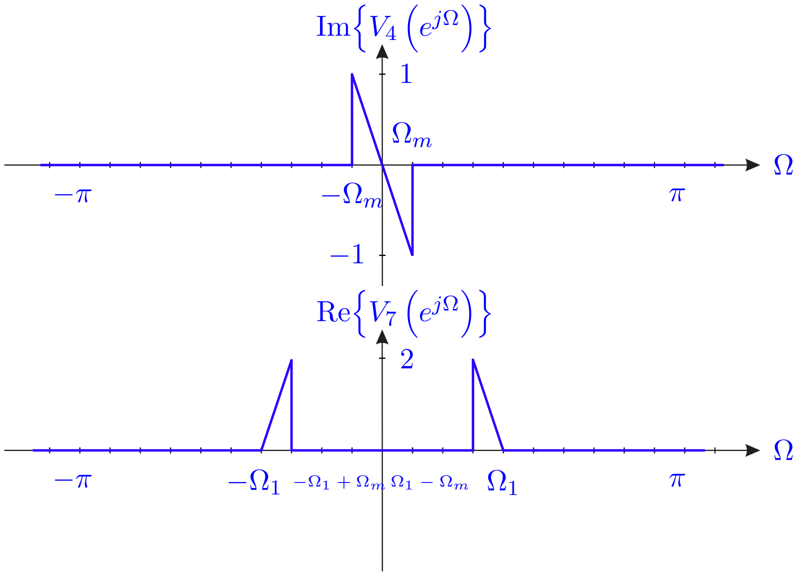
(c) Give the carrier frequency Ω_T. (1 P)
Carrier frequency $\Omega_T = \Omega_1$.

In the next step, a signal with the following spectrum is fed into the system:



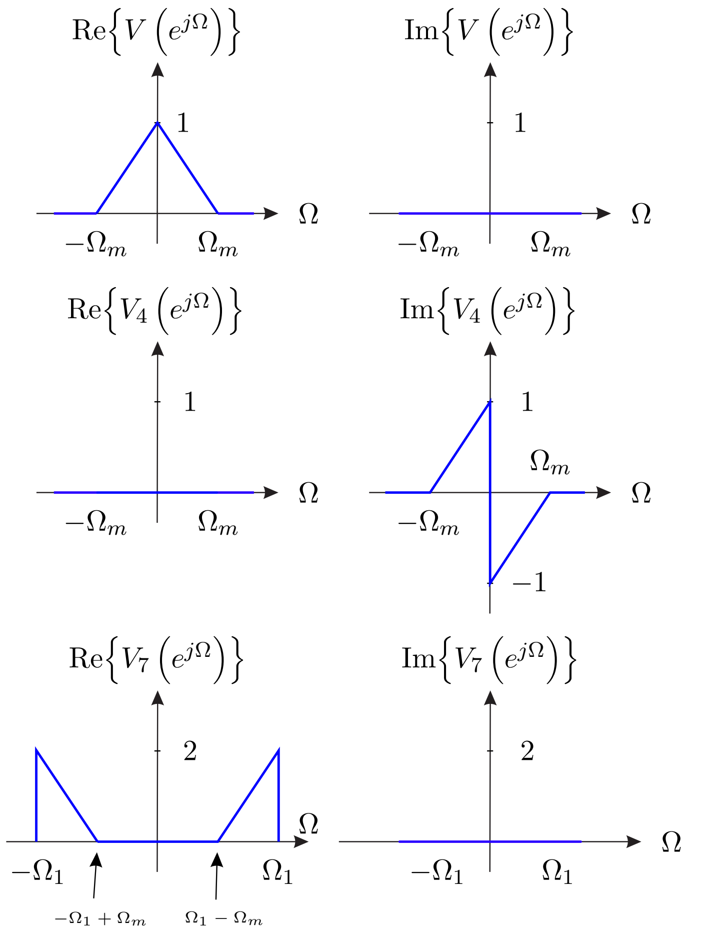
(d) Sketch both the real and imaginary part of the Fourier transforms $V_4\left(e^{j\Omega}\right)$ and (4 P)

$V_7\left(e^{j\Omega}\right)$ for $-\pi < \Omega < \pi$ with $\Omega_m = \frac{1}{10}\pi$, $\Omega_0 = \frac{1}{5}\pi$ and $\Omega_1 = 2 \cdot \Omega_0$. Label all axes.
$V_4\left(e^{j\Omega}\right)$ is purely imaginary, $V_7\left(e^{j\Omega}\right)$ purely real:



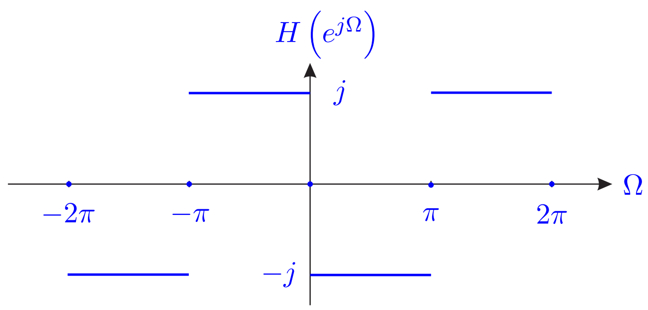
Part 2 *This part may be solved independently of part 1.*

Now the framed subsystem $S\{\cdots\}$ will be replaced by a new system $H\left(e^{j\Omega}\right)$. The following spectra are observed at the corresponding points of the circuit:



(e) Compare the input spectrum and the output spectrum. Which modulation type does the circuit apply? (2 P)
Single-sideband modulation using the lower sideband.

(f) Sketch the transfer function $H\left(e^{j\Omega}\right)$ for $-2\pi < \Omega < 2\pi$. Label all axes. What kind of transformation is it? (5 P)

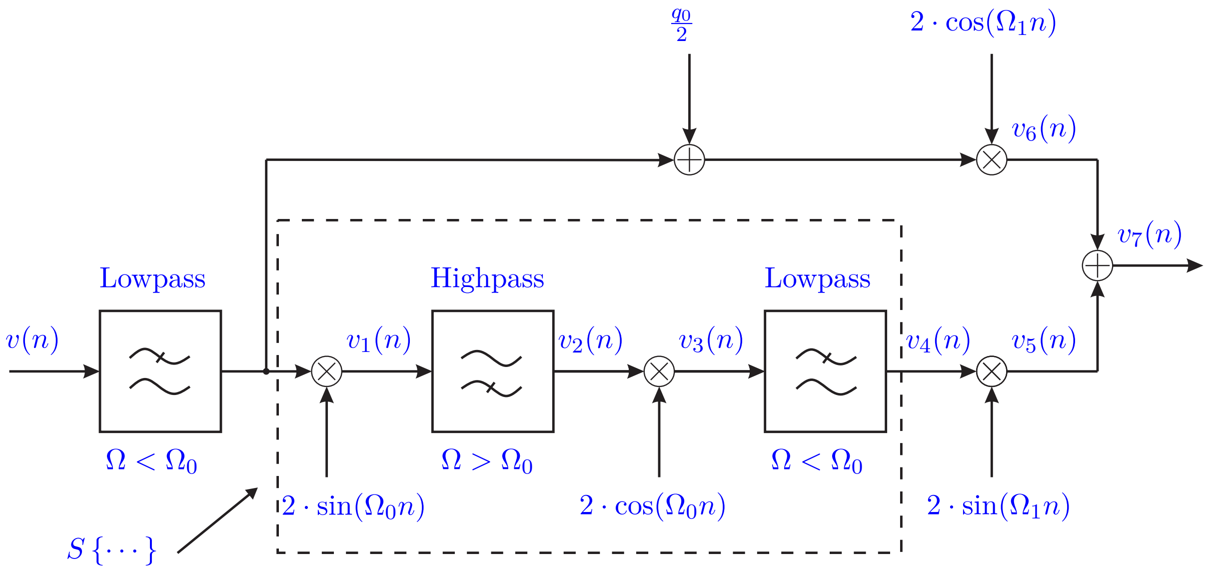


It is a Hilbert transform.

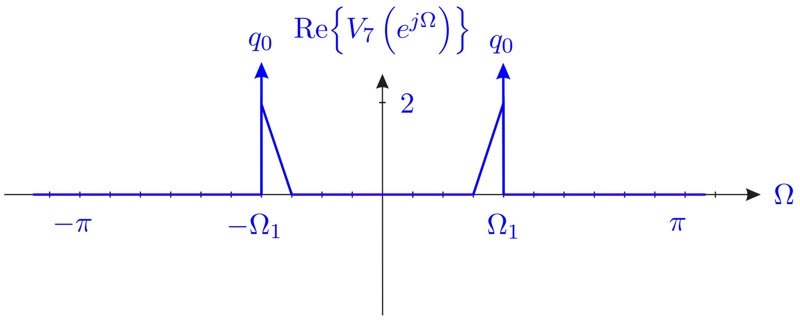
In order to simplify the demodulation, the information about the carrier signal should be added to the transmission signal $v_7(n)$.

(g) Expand the upper modulation system, so that the carrier signal is transmitted with (5 P)

the signal $v_7(n)$. Give the signal $v_7(n)$ in dependence of $v(n)$ and sketch the Fourier transform $V_7\left(e^{j\Omega}\right)$ for $-\pi < \Omega < \pi$ with $\Omega_m = \frac{1}{10}\pi$, $\Omega_0 = \frac{1}{5}\pi$ and $\Omega_1 = 2 \cdot \Omega_0$.



$$v_7(n) = \left(\frac{q_0}{2} + v(n)\right) \cdot 2 \cdot \cos(\Omega_1 n) + h(n) * v(n) \cdot 2 \cdot \sin(\Omega_1 n).$$



(h) Name at least two advantages of single-sideband modulation compared to double-sideband modulation. (2 P)

Half bandwidth requirement and higher range of the transmission signal through more efficient utilization of the transmission energy.

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