Advanced Signals and Systems

Exam WS 2016/2017

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	21.02.2017
Name:	______________________________
Matriculation Number:	______________________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorized to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ______________________________

Marking

Problem	1	2	3
Points	/42	/27	/31

Total number of points: ________________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated ________________ Signature: ______________________________

Advanced Signals and Systems

Exam WS 2016/2017

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	21.02.2017
Time:	09:00 h – 10:30 h (90 minutes)
Location:	KS2, C-SR II

Remarks

- Please write your **name** and your **matriculation number** on each sheet of paper that you return.
- Please keep your student ID and your identity card ready.
- During the exam only questions concerning the problems are answered.
- Please don't use any pencil or red pen.
- Please use a **new** sheet of paper with your name and matriculation number on it for **each problem**. You can ask for more sheets of paper, if necessary.
- The exam is open books, open notes; other people are closed. Programmable electronical devices except pocket calculators are not permitted.
- Partial credit will be given. No credit will be given if an answer appears with no supporting work or reason.
- Note that the given points of the subproblems are just preliminary.
- At the end of the exam put all sheets together as you have received them, including the problem sheets.
- No one is allowed to talk or to leave his or her seat until **all** exams have been collected.
- The problems and the solutions will be published on the website of the lecture. Also the date and the place of the inspection will be announced on this website.

Problem 1 (42 points)

Part I (*Can be solved individually from the other parts*)

Given the time discrete signals

$$\tilde{y}(n) = 2\gamma_{-1}(n) - 4\gamma_0(n-1) - 4\gamma_0(n-2) - 2\gamma_{-1}(n-4),$$

and

$$\tilde{z}(n) = \begin{cases} 1, & \text{for } n = 0, \\ -1, & \text{for } n = 1, \\ 0, & \text{else.} \end{cases}$$

Here $\gamma_{-1}(n)$ is the step sequence and $\gamma_0(n)$ is the unit impulse sequence. From the sequences $\tilde{y}(n)$ and $\tilde{z}(n)$ the sequences $y(n)$ and $z(n)$ respectively can be generated by extracting the signal values (of $\tilde{y}(n)$ and $\tilde{z}(n)$ respectively) at the indices $n = 0 \ldots 3$ and periodically repeating them (period length 4). Further more, it is given:

$$Z_4(\mu) = \text{DFT}_4\{z(n)\} = \begin{cases} 0 & \mu = 0, \\ 1+j & \mu = 1, \\ 2 & \mu = 2, \\ 1-j & \mu = 3. \end{cases}$$

(a) Determine $y(n)$ for $n \in \{0, \ldots, 3\}$. (1 P)

(b) Calculate the discrete Fourier transform $Y_4(\mu) = \text{DFT}_4\Big\{y(n)\Big\}$ of $y(n)$. (2 P)

$$Y_M(\mu) = \sum_{n=0}^{M-1} y(n) e^{-j\frac{2\pi}{M}n\mu}$$

$$Y_4(\mu) = \sum_{n=0}^{3} y(n) e^{-j\frac{\pi}{2}n\mu} = \underbrace{y(0)}_{=2} + \underbrace{y(1)}_{=-2} e^{-j\frac{\pi}{2}\mu} + \underbrace{y(2)}_{=-2} e^{-j\pi\mu} + \underbrace{y(3)}_{=2} e^{-j\frac{3\pi}{2}\mu}$$

$$Y_4(0) = y(0) + y(1) + y(2) + y(3) = 0$$
$$Y_4(1) = 2 - 2e^{-j\frac{\pi}{2}} - 2e^{-j\pi} + 2e^{-\frac{3\pi}{2}} = 4 + 4j$$
$$Y_4(2) = 2 - 2e^{-j\pi} - 2e^{-j2\pi} + 2e^{-j3\pi} = 0$$
$$Y_4(3) = 2 - 2e^{-j\frac{3\pi}{2}} - 2e^{-j\pi} + 2e^{-\frac{\pi}{2}} = 4 - 4j$$

(c) Calculate the output $\tilde{x}(n)$ of the linear convolution of $\tilde{y}(n)$ and $\tilde{z}(n)$ for indices $0 \le n \le 6$. (5 P)

$$\tilde{x}(n) = \tilde{y}(n) * \tilde{z}(n) = \sum_{k=-\infty}^{\infty} \tilde{y}(k)\tilde{z}(n-k)$$

$\tilde{y}(n)$ and $\tilde{z}(n)$ are non zero for $n \geq 0$.

$$\begin{aligned}
\tilde{x}(0) &= \sum_{k=0}^{\infty} \tilde{y}(k)\tilde{z}(-k) = \tilde{y}(0)\tilde{z}(0) + \tilde{y}(1)\tilde{z}(-1) + \dots \\
&= 2 \\
\tilde{x}(1) &= \sum_{k=0}^{\infty} \tilde{y}(k)\tilde{z}(1-k) = \tilde{y}(0)\tilde{z}(1) + \tilde{y}(1)\tilde{z}(0) + \tilde{y}(2)\tilde{z}(-1) + \dots \\
&= -4 \\
\tilde{x}(2) &= \sum_{k=0}^{\infty} \tilde{y}(k)\tilde{z}(2-k) = \tilde{y}(0)\tilde{z}(2) + \tilde{y}(1)\tilde{z}(1) + \tilde{y}(2)\tilde{z}(0) + \tilde{y}(3)\tilde{z}(-1) + \dots \\
&= 0 \\
\tilde{x}(3) &= \sum_{k=0}^{\infty} \tilde{y}(k)\tilde{z}(3-k) = \tilde{y}(0)\tilde{z}(3) + \tilde{y}(1)\tilde{z}(2) + \tilde{y}(2)\tilde{z}(1) + \tilde{y}(3)\tilde{z}(0) + \tilde{y}(4)\tilde{z}(-1) + \dots \\
&= 4 \\
\tilde{x}(4) &= \sum_{k=0}^{\infty} \tilde{y}(k)\tilde{z}(4-k) = \tilde{y}(0)\tilde{z}(4) + \tilde{y}(1)\tilde{z}(3) + \tilde{y}(2)\tilde{z}(2) + \tilde{y}(3)\tilde{z}(1) + \tilde{y}(4)\tilde{z}(0) + \dots \\
&= -2 \\
\tilde{x}(5) &= \sum_{k=0}^{\infty} \tilde{y}(k)\tilde{z}(5-k) = \tilde{y}(0)\tilde{z}(5) + \tilde{y}(1)\tilde{z}(4) + \tilde{y}(2)\tilde{z}(3) + \tilde{y}(3)\tilde{z}(2) + \tilde{y}(4)\tilde{z}(1) + \dots \\
&= 0 \\
\tilde{x}(6) &= \sum_{k=0}^{\infty} \tilde{y}(k)\tilde{z}(6-k) = \tilde{y}(0)\tilde{z}(6) + \tilde{y}(1)\tilde{z}(5) + \tilde{y}(2)\tilde{z}(4) + \tilde{y}(3)\tilde{z}(3) + \tilde{y}(4)\tilde{z}(2) + \dots \\
&= 0
\end{aligned}$$

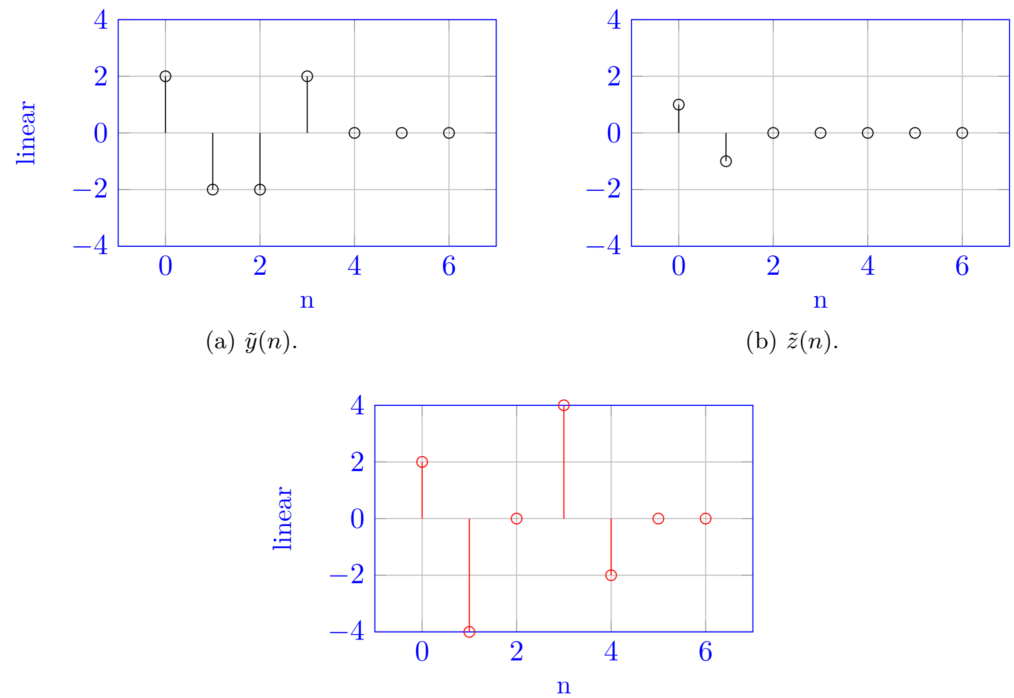
$$(M = M_1 + M_2 - 1)$$

(d) Sketch $\tilde{x}(n)$ from part **??** for $0 \leq n \leq 6$. (3 P)

See figure **??**.

(e) Determine $\mathrm{IDFT}_4\Big\{Y_4(\mu) \cdot Z_4(\mu)\Big\} = \mathrm{IDFT}_4\Big\{\mathrm{DFT}\left\{y(n)\right\} \cdot \mathrm{DFT}\left\{z(n)\right\}\Big\}$. (4 P)

$$\tilde{X}_4(\mu) = Y_4(\mu) \cdot Z_4(\mu) = \begin{cases} 0 & \mu \in \{0, 2\}, \\ 8j & \mu \in \{1\}, \\ -8j & \mu \in \{3\}, \end{cases}$$



(a) $\tilde{y}(n)$.

(b) $\tilde{z}(n)$.

(c) $\tilde{x}(n)$.

Figure 1: Signal plots.

$$\begin{aligned}
\tilde{x}(n) = \text{IDFT}\left\{\tilde{X}_4(\mu)\right\} &= \frac{1}{M}\sum_{\mu=0}^{M-1}\tilde{X}_M(\mu)e^{j\mu\frac{2\pi}{M}n} \\
&= \frac{1}{4}\left[\tilde{X}_4(0) + \tilde{X}_4(1)e^{j\frac{\pi}{2}n} + \tilde{X}_4(2)e^{j\pi n} + \tilde{X}_4(3)e^{j\frac{3\pi}{2}n}\right] \\
&= \frac{1}{4}\left[\tilde{X}_4(1)e^{j\frac{\pi}{2}n} + \tilde{X}_4(3)e^{j\frac{3\pi}{2}n}\right]
\end{aligned}$$

$$\begin{aligned}
\tilde{x}(0) &= \frac{1}{4}(8j - 8j) = 0 \\
\tilde{x}(1) &= \frac{1}{4}(8je^{j\frac{\pi}{2}} + -8je^{j\frac{3\pi}{2}}) = -4 \\
\tilde{x}(2) &= \frac{1}{4}(8je^{j\pi} + -8je^{j3\pi}) = 0 \\
\tilde{x}(3) &= \frac{1}{4}(8je^{j\frac{3\pi}{2}} + -8je^{j\frac{9\pi}{2}}) = 4
\end{aligned}$$

(f) Name an alternative calculation to **??** in the time domain. (2 P)

Cyclic convolution.

(g) Can the result from part **??** be directly calculated from the one in part **??**? Justify your answer! (2 P)

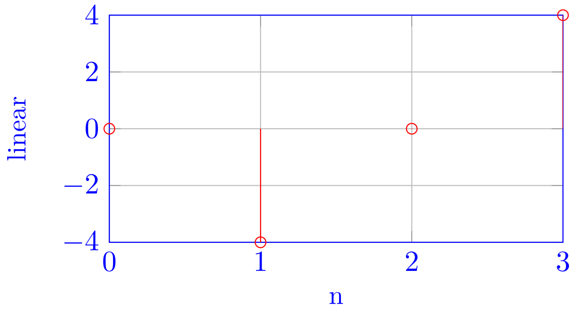


Figure 2: $\tilde{x}(n) = y(n) \circledast z(n)$.

Yes, the result of the cyclic convultion of $\tilde{x}(n)$ with $\tilde{M} = 4$ yields from the linear convolution ($x(n)$) with length M by following the below named scheme:

$$\tilde{x}(n) = \sum_{l=-\infty}^{\infty} x(n + lM).$$

For $x(n)$ zeropadding needs to be applied for $n \geq M$.

$$\begin{aligned} \tilde{x}(0) &= x(0) + x(4) = 0 \\ \tilde{x}(1) &= x(1) + x(5) = -4 \\ \tilde{x}(2) &= x(2) + x(6) = 0 \\ \tilde{x}(3) &= x(3) + x(7) = 4 \end{aligned}$$

Part II (*Can be solved individually from the other parts*)

Given the time discrete Fourier transform of $v(n)$:

$$V(e^{j\Omega}) = \frac{\sin^2\left(\frac{2N+1}{2}\Omega\right)}{\sin^2\left(\frac{\Omega}{2}\right)}.$$

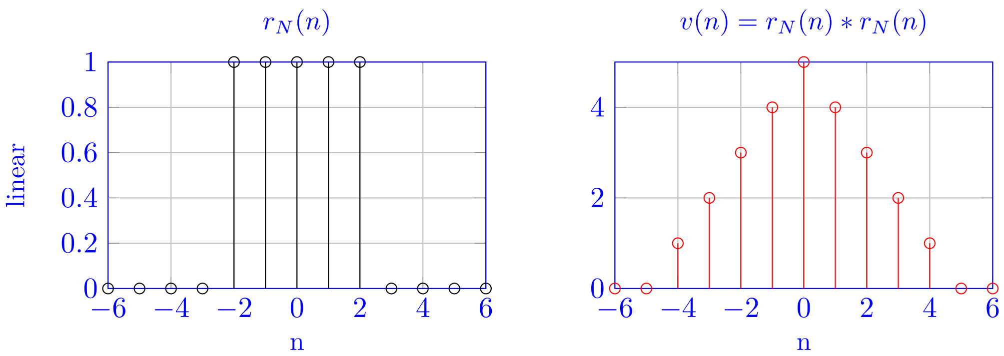
(h) Calculate the inverse time discrete Fourier transform $v(n) = \mathcal{F}^{-1}\left\{V(e^{j\Omega})\right\}$. (5 P)

$$\begin{aligned} v(n) &= \mathcal{F}^{-1}\left\{\frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)} \cdot \frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)}\right\} \\ &= \text{rect}_N(n) * \text{rect}_N(n) \\ &= d_{2N}(n) \end{aligned}$$

with:

$$\text{rect}_N = \begin{cases} 1, & -N \geq n \leq N \\ 0, & \text{else.} \end{cases}$$

(i) Sketch $v(n)$ for $N = 2$ for $-6 \leq n \leq 6$. (3 P)



(a) Rectangular function. (b) Solution of task.

Part III (*Can be solved individually from the other parts*)
Now the frequency response:

$$H(z) = \frac{z^2 - \frac{1}{4}}{z^3 - \frac{1}{4}z^2 + \frac{1}{2}jz - \frac{1}{8}j}$$

is given.

(j) Determine the poles and zeros of the system. (6 P)
Hint: $\sqrt{j} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$, one pole is at $z_{\infty,0} = \frac{1}{4}$.
Calculating the zeros:
It can be seen that $z^2 - \frac{1}{4}$ has the form of the third binomial equation which has the form:

$$z^2 - \frac{1}{4} = \left(z + \frac{1}{2}\right)\left(z - \frac{1}{2}\right).$$

Such the zeros are $z_{0,0} = \frac{1}{2}$ and $z_{0,1} = -\frac{1}{2}$. Alternatively the p/q formula

$$z_{0,(0,1)} = p/2 \pm \sqrt{\left(\frac{p}{2}\right)^2 - q},$$

can be used.

Calculating the poles:
The first pole is given in the hint and equals $z_{\infty,0} = \frac{1}{4}$.
Extracting the known pole by division yields:

$$z^3 - \frac{1}{4}z^2 + \frac{1}{2}jz - \frac{1}{8}j \;:\; \left(z - \frac{1}{4}\right) = z^2 + \frac{1}{2}j.$$

Now the p/q formula and the given hint ($\sqrt{j} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$) can be used to determine the remaining poles:

$$z_{\infty,1} = \frac{1}{2} - \frac{1}{2}j,$$
$$z_{\infty,2} = -\frac{1}{2} + \frac{1}{2}j.$$

(k) Sketch the pole-zero plot. (3 P)

See fig. **??**.

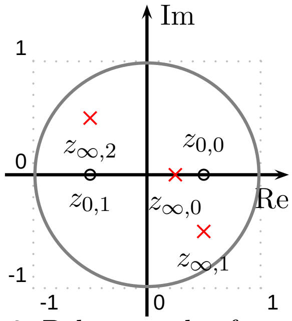


Figure 3: Pole zero plot for task *(j)*.

(l) Sketch the region of convergence in the pole-zero plot. (3 P)

ROC is region outside the outmost pole in case of causal systems, i.e. $|z| > |z_{\infty,(1,2)}|$. See fig. **??**.

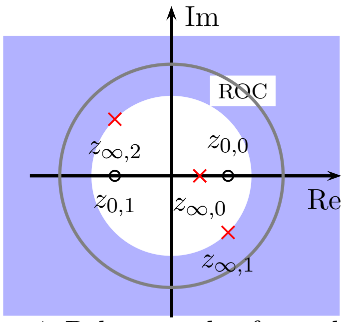


Figure 4: Pole zero plot for task *(k)*.

(m) Add a minimum number of poles and/or zeros such that the system gets real valued. (1 P)

First possibility: Add zeros $z_{0,2} = \frac{1}{2} - \frac{1}{2}j$ and $z_{0,3} = -\frac{1}{2} + \frac{1}{2}j$ to get a real valued system such the poles $z_{\infty,1}$ and $z_{\infty,2}$ are crossed out and just $\{z_{\infty,0}, z_{0,0}, z_{0,1}\} \in \mathcal{R}$ are left.

Second possibility: Add poles $z_{\infty,4} = \frac{1}{2} + \frac{1}{2}j$ and $z_{\infty,5} = -\frac{1}{2} - \frac{1}{2}j$ to get a real valued system.

(n) Is the new system from part *(m)* causal? Justify your answer. (2 P)

First possibility: No as there are less poles than zeros now. (Optional: Can be proven by determining the difference equation.)

Second possibility: Yes as there are more poles than zeros.

(Optional: Can be proven by determining the difference equation.)

Problem 2 (27 points)

Given is the autocorrelation function of the signal $v(n)$ with $A \in \mathbb{R}$.

$$s_{vv}(\kappa) = A^{\kappa}, \text{ for } \kappa \geq 0 \ .$$

(a) Find $s_{vv}(\kappa)$ for $\kappa < 0$. And give the final function $s_{vv}(\kappa)$ $\kappa \in [-\infty, \infty]$. (4 P)

$$\begin{aligned} s_{vv}(-\kappa) &= s^*_{vv}(\kappa) \\ s_{vv}(\kappa) &= A^{-\kappa}, \text{for } \kappa < 0 \\ s_{vv}(\kappa) &= A^{|\kappa|} \end{aligned}$$

(b) Calculate the mean and the variance of $v(n)$. Assume $0 < |A| < 1$. (4 P)

$$\begin{aligned} \mu_v &= \sqrt{\lim_{\kappa \to \infty} s_{vv}(\kappa)} \\ &= \sqrt{\lim_{\kappa \to \infty} A^{|\kappa|}} \\ &= 0 \\ Var\{v(n)\} &= \sigma_v^2 = E\left\{(v - E\{v\})^2\right\} \\ &= E\left\{(v)^2\right\} \\ &= s_{vv}(0) = A^0 = 1 \end{aligned}$$

(c) Calculate the z-Transform $S_{vv}(z)$ of $s_{vv}(n)$. (8 P)

$$\begin{aligned} S_{vv}(z) &= \sum_{n=-\infty}^{\infty} s_{vv}(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} A^{|n|} z^{-n} \\ &= \sum_{n=-\infty}^{0} A^{-n} z^{-n} + \sum_{n=0}^{\infty} A^{n} z^{-n} - 1 \\ &= \sum_{n=0}^{\infty} (Az)^n + \sum_{n=0}^{\infty} \left(\frac{A}{z}\right)^n - 1 \\ &= \frac{1}{1 - Az} + \frac{1}{1 - \frac{A}{z}} - 1 \text{ ,for} A < |z| < \frac{1}{A} \end{aligned}$$

(d) Find a value for A to get the following ROC (Region of Convergance) $2 > |z| > \frac{1}{2}$. (3 P)

$$|Az| < 1 \rightarrow |z| < \frac{1}{A}$$
$$\left|\frac{A}{z}\right| < 1 \rightarrow |z| > A$$
$$A = \frac{1}{2}$$

(e) Now a system with the following tansfer function (8 P)

$$H(z) = \frac{z+3}{z+\frac{1}{3}}$$

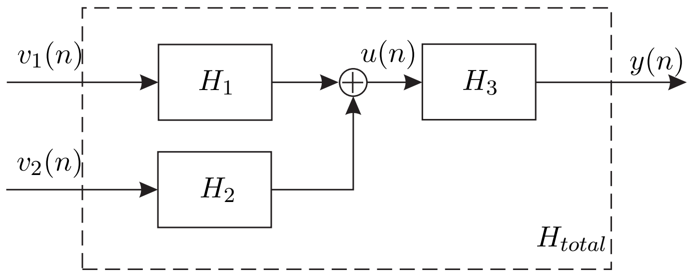
is applied to the input signal $v(n)$. Calculate $S_{yy}(z)$ and $s_{yy}(\kappa)$ of the output $y(n)$.

Hint: If a system only consists of real valued poles and zeros, following simplification can be used: $H^*\left(\frac{1}{z^*}\right) = H\left(\frac{1}{z}\right)$

$$\begin{aligned}
S_{yy}(z) &= S_{vv}(z) \cdot H(z) \cdot H^*\left(\frac{1}{z^*}\right) \\
&= S_{vv}(z) \frac{\frac{1}{z}+3}{\frac{1}{z}+\frac{1}{3}} \cdot \frac{z+3}{z+\frac{1}{3}} \\
&= S_{vv}(z) \frac{1+3z}{1+\frac{z}{3}} \cdot \frac{z+3}{z+\frac{1}{3}} \\
&= S_{vv}(z) \frac{3(\frac{1}{3}+z)}{1+\frac{z}{3}} \cdot \frac{3(\frac{z}{3}+1)}{z+\frac{1}{3}} \\
&= S_{vv}(z) \cdot 9 \rightarrow s_{yy}(\kappa) = s_{vv}(\kappa) \cdot 9
\end{aligned}$$

Problem 3 (31 points)

Given is a discrete system according to the following block diagram.



For the systems H_1 and H_2 the following transfer functions hold:

$$H_1(z) = z - 1,$$
$$H_2(z) = z + 1.$$

The system H_3 can be described by the following difference equation:

$$y(n) = u(n) + \frac{1}{2}y(n-1) + \frac{1}{2}y(n-2)$$

(a) Give a statement about the characteristics (highpass, lowpass, bandpass, ...) of the system $H_1(z)$. Do the same for the system $H_2(z)$. Give reason to your answers. (4 P)

The system H_1 is a highpass, because the only zero $z = 1$ of the system is at frequency zero and the system has no poles.
The system H_2 is a lowpass, because the only zero of the system $z = -1$ is at half the sample frequency and the system has no poles.

(b) Determine the transfer function $H_3(z)$. (2 P)

Z-Transformation of the difference equation for $h_3(n)$ leads to:

$$Y(z) = U(z) + \frac{1}{2}z^{-1}Y(z) + \frac{1}{2}z^{-2}Y(z)$$
$$\left(1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}\right)Y(z) = U(z)$$

For the transfer function it follows $H_3(z) = \frac{Y(z)}{U(z)}$:

$$H_3(z) = \frac{1}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}$$
$$= \frac{z^2}{z^2 - \frac{1}{2}z - \frac{1}{2}}$$

From now on the following transfer functions hold for the systems $H_1(z)$ and $H_2(z)$:

$$H_1(z) = \frac{z-\alpha}{z},$$
$$H_2(z) = \frac{z+1}{z}.$$

Preliminarily, for the following problems $\alpha = 1$ holds till task (h).

(c) Calculate the transfer matrix for the total system $\mathbf{H}_{total}(z)$. Simplify the expression as much as possible. (3 P)

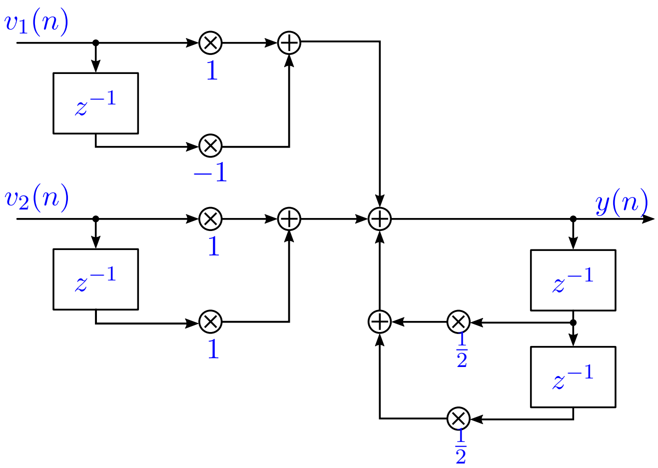
$$\mathbf{H}_{total}(z) = \begin{bmatrix} H_1(z) \cdot H_3(z) \\ H_2(z) \cdot H_3(z) \end{bmatrix} = \begin{bmatrix} \frac{z^2(z-1)}{z\left(z^2-\frac{1}{2}z-\frac{1}{2}\right)} \\ \frac{z^2(z+1)}{z\left(z^2-\frac{1}{2}z-\frac{1}{2}\right)} \end{bmatrix} = \begin{bmatrix} \frac{z(z-1)}{(z-1)\left(z+\frac{1}{2}\right)} \\ \frac{z(z+1)}{(z-1)\left(z+\frac{1}{2}\right)} \end{bmatrix} = \begin{bmatrix} \frac{z}{\left(z+\frac{1}{2}\right)} \\ \frac{z^2+z}{(z-1)\left(z+\frac{1}{2}\right)} \end{bmatrix}$$

(d) Give the number of inputs, outputs and the minimum number of internal states of the total system H_{total}? (2 P)

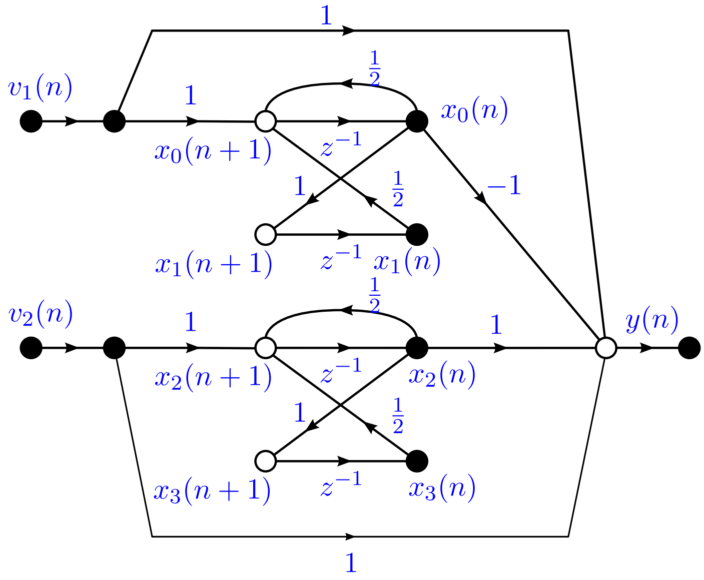
Inputs: $L = 2$
Outputs: $R = 1$
States: $N = 4$

(e) Sketch the signal flow chart of the state space description of the total system H_{total}. (7 P)

Determine the direct form of the signal flow chart directly from the block diagram and the transfer function:



From the direct form, the signal flow chart can be obtained:



(f) Does the total system H_{total} have a direct pass through? Give reason to your answer. (2 P)

Yes, both inputs have a direct connection to the output of the system as can be seen in the signal flow graph.

(g) Now the system should be described in the state space by using the state- and the output equation (6 P)

$$\mathbf{x}(n+1) = \mathbf{A}\,\mathbf{x}(n) + \mathbf{B}\,\mathbf{v}(n)$$
$$\mathbf{y}(n) = \mathbf{C}\,\mathbf{x}(n) + \mathbf{D}\,\mathbf{v}(n)$$

Where $\mathbf{v}(n)$ describes the inputs, $\mathbf{y}(n)$ the outputs and $\mathbf{x}(n)$ the internal states of the system. Determine the matrices (respectively vectors and scalars) $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$.

The matrices and vectors can directly be read from the signal flow chart:

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{C} = \mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}^{\mathrm{T}} \quad \mathbf{D} = \mathbf{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\mathrm{T}}$$

For the following problems the parameter α of the system H_1 be variable.

(h) Does a change of the parameter α have an impact on the direct pass through of the total system H_{total}? Explain how the pass through changes in dependency of α. (2 P)

No, since α is the weight corresponding to the delayed input.

(i) Does a change of the parameter α have an impact on the characteristics (bandpass, highpass, lowpass) of the system H_1? Explain the change in the characteristics of H_1 in dependency of α. (3 P)

Yes, when changing the parameter α the zero of the system changes, which has an impact on the characteristics of the system H_1. For the case $\alpha > 0$ the system H_1 stays a highpass. For $\alpha = 0$ the System becomes an allpass. And for the case $\alpha < 0$ the system H_1 is a lowpass filter.