Advanced Signals and Systems

Exam WS 2016/2017

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	26.09.2017
Name:	______________________________
Matriculation Number:	______________________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorized to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ______________________________

Marking

Problem	1	2	3
Points	/33	/26	/41

Total number of points: ________________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated ________________ Signature: ______________________________

Advanced Signals and Systems

Exam WS 2016/2017

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	26.09.2017
Time:	09:00 h – 10:30 h (90 minutes)
Location:	KS2, C-SR II

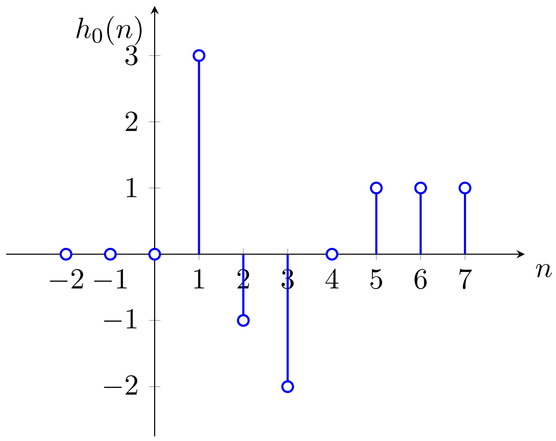
Remarks

- Please write your **name** and your **matriculation number** on each sheet of paper that you return.
- Please keep your student ID and your identity card ready.
- During the exam only questions concerning the problems are answered.
- Please don't use any pencil or red pen.
- Please use a **new** sheet of paper with your name and matriculation number on it for **each problem**. You can ask for more sheets of paper, if necessary.
- The exam is open books, open notes; other people are closed. Programmable electronical devices except pocket calculators are not permitted.
- Partial credit will be given. No credit will be given if an answer appears with no supporting work or reason.
- Note that the given points of the subproblems are just preliminary.
- At the end of the exam put all sheets together as you have received them, including the problem sheets.
- No one is allowed to talk or to leave his or her seat until **all** exams have been collected.
- The problems and the solutions will be published on the website of the lecture. Also the date and the place of the inspection will be announced on this website.

Problem 1 (33 points)

Part 1 *This part may be solved independently of Part 2.*

Given is the discrete impulse response $h_0(n)$:



In addition to the graph the following holds:

$$h_0(n) = 0 \ \forall \ n < 0,$$
$$h_0(n) = 1 \ \forall \ n \geq 5.$$

(a) Determine the equation for the impulse response based on weighted impulse- and step-functions. (3 P)

$$h_0(n) = 3\gamma_0(n-1) - \gamma_0(n-2) - 2\gamma_0(n-3) + \gamma_{-1}(n-5)$$

(b) Does the system have a direct passthrough? Give reason to your answer. (2 P)
(*Hint:* No calculations necessary.)
No passthrough because $h_0(0) = 0$.

(c) Determine the transfer function $H(z)$. (3 P)
Using known correspondences one obtains:

$$\begin{aligned} H(z) &= 3z^{-1} - z^{-2} - 2z^{-3} + z^{-5}\frac{z}{z-1} \\ &= 3z^{-1} - z^{-2} - 2z^{-3} + z^{-4}\frac{1}{z-1} \end{aligned}$$

(d) Determine the difference equation. (5 P)

$$\begin{aligned} H(z) &= \frac{Y(z)}{V(z)} \\ &= 3z^{-1} - z^{-2} - 2z^{-3} + z^{-4}\frac{1}{z-1} \\ &= \frac{3 - 4z^{-1} - z^{-2} + 2z^{-3} + z^{-4}}{z-1}, \end{aligned}$$

it follows:

$$
\begin{aligned}
(z-1)Y(z) &= (3 - 4z^{-1} - z^{-2} + 2z^{-3} + z^{-4})V(z) \\
Y(z) &= (3z^{-1} - 4z^{-2} - z^{-3} + 2z^{-4} + z^{-5})V(z) + z^{-1}Y(z)
\end{aligned}
$$

Transformation to the time domain yields the difference equation:

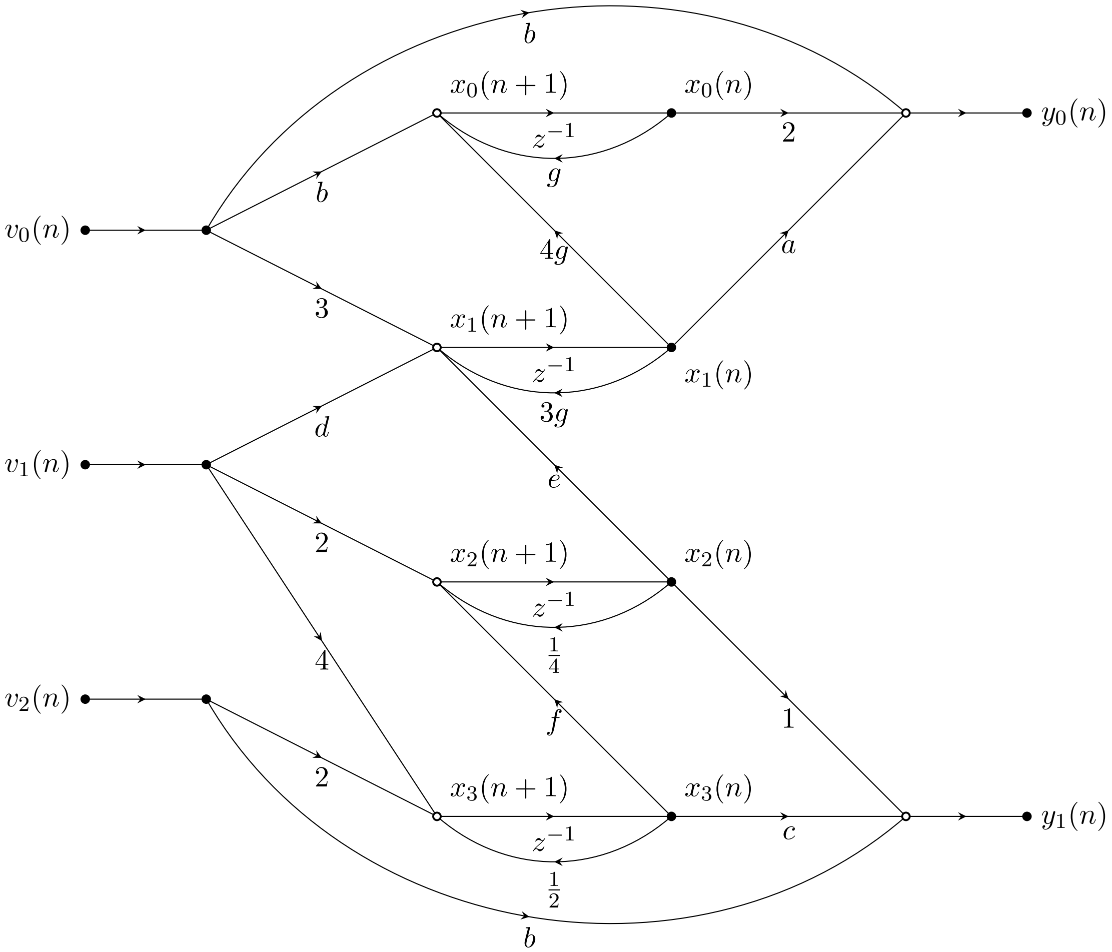
$$
y(n) = 3v(n-1) - 4v(n-2) - v(n-3) + 2v(n-4) + v(n-5) + y(n-1)
$$

Part 2 *This part may be solved independently of Part 1.*

In general a system in state-space description can be described by:

$$\mathbf{x}(n+1) = \mathbf{A}\,\mathbf{x}(n) + \mathbf{B}\,\mathbf{v}(n)$$
$$\mathbf{y}(n) = \mathbf{C}\,\mathbf{x}(n) + \mathbf{D}\,\mathbf{v}(n)$$

Given is the signal-flow graph with $a, b, c, d, e, f, g \in \mathbb{R}$:



Hint:: Eigenvalues of triangular matrices are the elements of the main diagonal.

(e) How many inputs, outputs and states does the system have? Which dimensions do the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ have? (4 P)

$N = 4$ states, $L = 3$ inputs, $R = 2$ outputs.

$$\mathbf{A} \rightarrow [N\text{x}N] = [4\text{x}4]$$
$$\mathbf{B} \rightarrow [N\text{x}L] = [4\text{x}3]$$
$$\mathbf{C} \rightarrow [R\text{x}N] = [2\text{x}4]$$
$$\mathbf{D} \rightarrow [R\text{x}L] = [2\text{x}3]$$

(f) Determine the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ and assign them their common names. (6 P)

From the signal-flow graph it directly follows:

$$\mathbf{A} = \begin{bmatrix} g & 4g & 0 & 0 \\ 0 & 3g & e & 0 \\ 0 & 0 & \frac{1}{4} & f \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b & 0 & 0 \\ 3 & d & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & a & 0 & 0 \\ 0 & 0 & 1 & c \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & b \end{bmatrix}$$

$\mathbf{A}$ system matrix, $\mathbf{B}$ steering matrix, $\mathbf{C}$ observation matrix, $\mathbf{D}$ pass-through matrix.

(g) Determine the characteristic polynomial of the system matrix. (4 P)

$$N(z) = |z\mathbf{E} - \mathbf{A}| = \begin{vmatrix} z-g & -4g & 0 & 0 \\ 0 & z-3g & -e & 0 \\ 0 & 0 & z-\frac{1}{4} & -f \\ 0 & 0 & 0 & z-\frac{1}{2} \end{vmatrix} = (z-g)(z-3g)\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)$$

$$= \left(z^2 - 4gz + 3g^2\right)\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right)$$

$$= z^4 - \frac{3}{4}z^3 + \frac{1}{8}z^2 - 4gz^3 + 3gz^2 - \frac{1}{2}gz + 3g^2z^2 - \frac{9}{4}g^2z + \frac{3}{8}g^2$$

(h) How can the characteristic polynomial be interpreted with regard to the transfer function? Give reason to your answer. (2 P)

The characteristic polynomial is equal to the denominator of the transfer function. Due to this the stability of the system is characterized by the zeros of the characteristic polynomial (i.e. the eigenvalues of matrix $\mathbf{A}$, which correspond to the poles).

(i) In which ranges do you need to choose the factors a, b, c, d, e, f, g such that the system is stable? (4 P)

Stability is given if $|z| < 1$ holds for all poles (i.e. poles lie within the unit circle). a, b, c, d, e, f have no influence on stability and such can be chosen arbitrarily:

$$a, b, c, d, e, f \in \mathbb{R}.$$

The poles of the characteristic polynomial are given by (see solution of *(g)*, poles are eigenvalues of matrix $\mathbf{A}$):

$$z_{\infty,0} = g,\ z_{\infty,1} = 3g,\ z_{\infty,2} = \frac{1}{4},\ z_{\infty,3} = \frac{1}{2}.$$

Because $|z_{\infty,i}| < 1\ \forall i \in [0, \ldots, 3]$ should hold it follows:

$$\max_{i \in [0,\ldots,3]} \{|z_{\infty,i}|\} < 1,$$

with $|z_{\infty,2,3}| < 1$ we demand:

$$|3g| < 1,$$
$$\Leftrightarrow |g| < \frac{1}{3}.$$

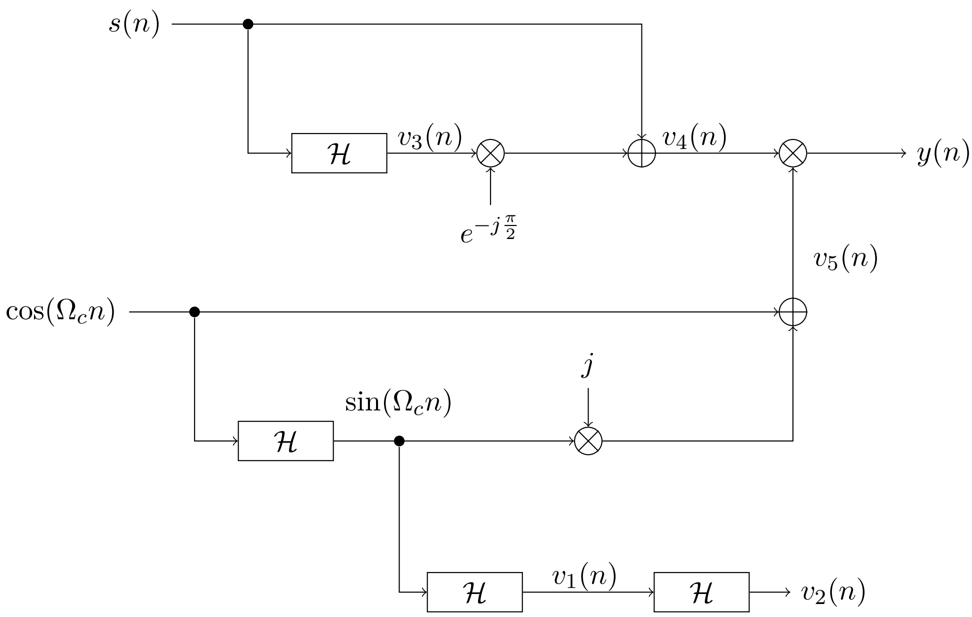
Because the task restricts the factors to real numbers it follows:

$$-\frac{1}{3} < g < \frac{1}{3}$$

Problem 2 (26 points)

Part 1 *This part may be solved independently of Part 2.*

Given is the following block diagram:



The subsystem $\mathcal{H}$ is the ideal Hilbert transformation. The signal $s(n)$ is fully defined by its spectrum $S(e^{j\Omega})$. Since the spectrum is periodic, only the range $\Omega \in [-\pi, \pi)$ needs to be considered in all tasks. For this range the following definition holds:

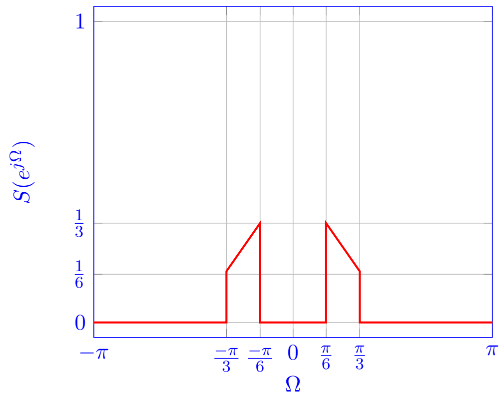
$$S(e^{j\Omega}) = \begin{cases} \frac{1}{2}\frac{-2|\Omega|+\pi}{\pi}, & \text{for } \frac{\pi}{6} \leq |\Omega| \leq \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

(a) Compute $v_1(n)$ and $v_2(n)$. (2 P)

$v_1(n) = -\cos(\Omega_c n)$

$v_2(n) = -\sin(\Omega_c n)$

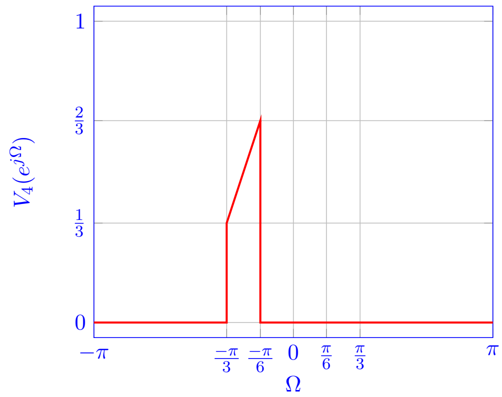
(b) Sketch $S(e^{j\Omega})$ in the interval $\Omega \in [-\pi, \pi)$. (3 P)



(c) Compute and sketch the spectrum of $v_4(n)$ in the interval $\Omega \in [-\pi, \pi)$. (5 P)

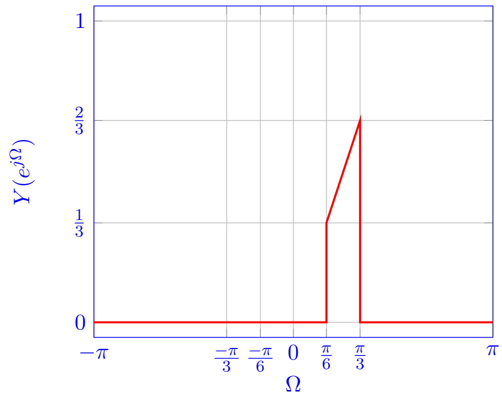
$$\begin{aligned} V_4(e^{j\Omega}) &= S(e^{j\Omega}) + j^2 \mathrm{sgn}(\Omega) S(e^{j\Omega}) \\ &= S(e^{j\Omega})(1 - \mathrm{sgn}(\Omega)) \end{aligned}$$

$$V_4(e^{j\Omega}) = \begin{cases} 0S(e^{j\Omega}), & \text{for } \Omega \geq 0 \\ 2S(e^{j\Omega}), & \text{for } \Omega < 0 \end{cases}$$



(d) Compute the spectrum of the output $y(n)$ with $\Omega_c = \frac{\pi}{2}$. (5 P)
The lower branch yields $\cos(\Omega_c n) + j\sin(\Omega_c n) = e^{j\Omega_c n}$. Consequently, the spectrum of $y(n)$ is the spectrum of $v_4(n)$ shifted by $\frac{\pi}{2}$ to the right.

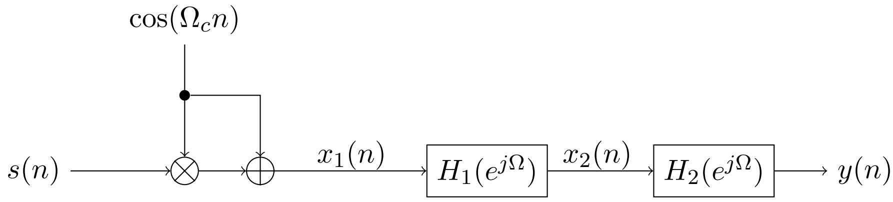
$$Y(e^{j\Omega}) = \begin{cases} \frac{2}{\pi}\Omega, & \text{for } \frac{\pi}{6} \leq \Omega \leq \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$



(e) Which modulation type is implemented by the system regarding the input $s(n)$ and the output $y(n)$? (2 P)
single-sideband modulation (lower side-band)

Part 2 *This part may be solved independently of Part 1.*

Given is the following block diagram:



The signal $s(n)$ is fully defined by its spectrum $S(e^{j\Omega})$. Since the spectrum is periodic, only the range $\Omega \in [-\pi, \pi)$ needs to be considered in all tasks. For this range the following definitions hold:

$$S(e^{j\Omega}) = \begin{cases} |\Omega|, & \text{for } \frac{\pi}{12} \leq |\Omega| \leq \frac{\pi}{6} \\ 0, & \text{else} \end{cases}$$

$$H_1(e^{j\Omega}) = \begin{cases} 0, & \text{for } 0 \leq \Big| |\Omega| - \Omega_c \Big| \leq \frac{\pi}{15} \\ 1, & \text{else} \end{cases}$$

(f) What is the modulation type regarding the following signals? Give reason to your answer! (4 P)

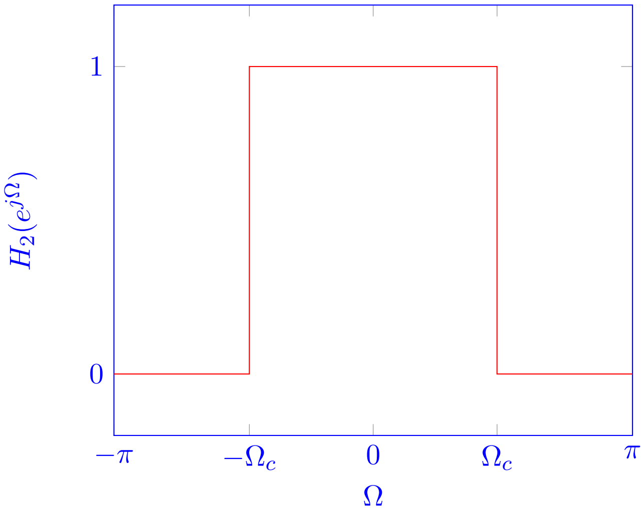
(i) $x_1(n)$
amplitude modulation, since a real signal is upmixed and the carrier is, additionally, added.

(ii) $x_2(n)$
amplitude modulation without carrier, since $H_1(e^{j\Omega})$ removes the carrier-> double-sideband modulation

(g) Give both a formula and a sketch for $H_2(e^{j\Omega})$ such that $y(n)$ is a single-sideband modulation of $s(n)$. (5 P)
Both a high- and a lowpass with the approximated cutoff frequency Ω_c are good candidates for $H_2(e^{j\Omega})$.

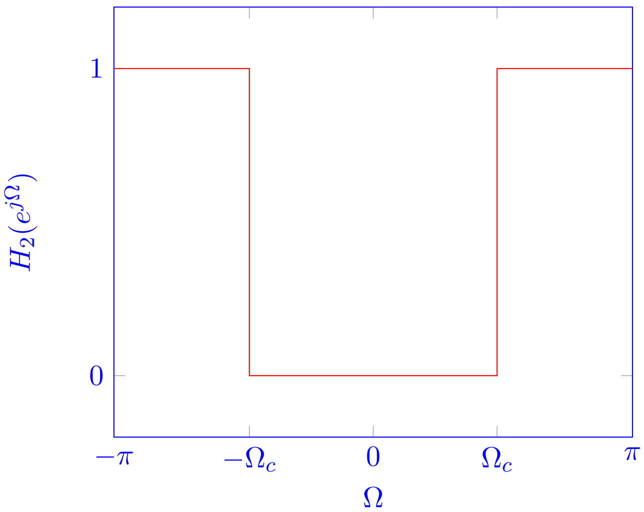
LP:

$$H_2(e^{j\Omega}) = \begin{cases} 1, & \text{for } 0 \leq |\ \Omega\ | \leq \Omega_c \\ 0, & \text{else} \end{cases}$$



HP:

$$H_2(e^{j\Omega}) = \begin{cases} 0, & \text{for } 0 \leq |\ \Omega\ | \leq \Omega_c \\ 1, & \text{else} \end{cases}$$



Problem 3 (41 points)

Given are the stochastic signals $x(t, \phi_1)$ and $y(t, \phi_2)$ with

$$x(t, \phi_1) = e^{j(\omega(t)+\phi_1)}$$
$$y(t, \phi_2) = e^{-j(\omega(t)+\phi_2)}.$$

For all time instances $t \in (-\infty, \infty)$ the definition $\omega(t) = 4t$ holds. The two processes $\phi_{1,2}$ are uniformly distributed over $[\pi, \gamma)$ with $\gamma > \pi$. Assume ϕ_1 and ϕ_2 to be statistically independent of each other.

Part 1 *This part may be solved independently of Part 2 and 3.*

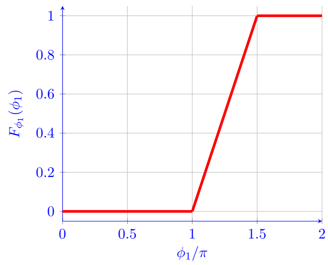
(a) Are ϕ_1 and ϕ_2 correlated? Give reason to your answer! (2 P)
ϕ_1 and ϕ_2 are uncorrelated since they are statistically independent.

(b) Give the probability density function $f_{\phi_1\phi_2}(\phi_1, \phi_2)$. (2 P)

$$f_{\phi_1\phi_2}(\phi_1, \phi_2) = \begin{cases} \frac{1}{(\gamma-\pi)^2}, & \text{for } \pi \le \phi_1 < \gamma \quad \wedge \quad \pi \le \phi_2 < \gamma \\ 0, & \text{else} \end{cases}$$

(c) Calculate the cumulative distribution function $F_{\phi_1}(\phi_1)$ and sketch it in the interval $[0, 2\pi]$ with all relevant information. Assume $\gamma = \frac{3\pi}{2}$. (4 P)

$$F_{\phi_1}(\phi_1) = \begin{cases} 0, & \text{for } \phi_1 < \pi \\ \frac{2}{\pi}(\phi_1 - \pi), & \text{for } \pi \le \phi_1 < \frac{3\pi}{2} \\ 1, & \text{for } \frac{3\pi}{2} \le \phi_1 \end{cases}$$



Part 2 *This part may be solved independently of Part 1 and 3.*

With $\phi_1 \equiv \phi_2$ we obtain

$$z(t, \phi_1) = x(t, \phi_1) + y(t, \phi_1).$$

(d) Simplify the expression for $z(t, \phi_1)$. How may this signal be interpreted? (3 P)
The signal is a scaled cosine with random phase.

$$\begin{aligned} z(t,\phi_1) &= e^{j(\omega(t)+\phi_1)} + e^{-j(\omega(t)+\phi_1)} \\ &= 2\cos(\omega(t)+\phi_1) \end{aligned}$$

(e) Calculate the second statistical moment regarding $z(t, \phi_1)$ in dependance of $\omega(t)$ and γ. **Hint**: $\cos^2(\theta) = \frac{1}{2}\cos(2\theta) + \frac{1}{2}$ (5 P)
The second moment is defined by

$$m_z^{(2)} = \mathrm{E}\left\{z^2(t,\phi_1)\right\}$$

Thus, the moment may be calculated as

$$\begin{aligned} m_z^{(2)}(t) &= 4\int_{-\infty}^{\infty} \cos^2(\omega(t)+\phi_1) f_{\phi_1}(\phi_1)\mathrm{d}\phi_1 = \frac{4}{\gamma-\pi}\int_{\pi}^{\gamma} \cos^2(\omega(t)+\phi_1)\mathrm{d}\phi_1 \\ &= \frac{4}{\gamma-\pi}\int_{\pi}^{\gamma}\left(\frac{1}{2}\cos(2\omega(t)+2\phi_1)+\frac{1}{2}\right)\mathrm{d}\phi_1 = \frac{2}{\gamma-\pi}\left[\frac{1}{2}\sin(2\omega(t)+2\phi_1)+\phi_1\right]_{\pi}^{\gamma} \\ &= 2 + \frac{\sin(2\omega(t)+2\gamma)-\sin(2\omega(t)+2\pi)}{\gamma-\pi} \end{aligned}$$

(f) Derive a rule for the values which γ is allowed to take such that the signal $z(t, \phi_1)$ is stationary regarding the second statistical moment. Use the result of task *(e)* as a starting point. (3 P)
From *(e)* it is obvious that the time dependence of the second moment vanishes if

$$\sin(2\omega(t)+2\gamma) = \sin(2\omega(t)+2\pi)$$

holds. From this follows

$$\begin{aligned} 2\gamma &= 2\pi + k2\pi \\ \Leftrightarrow \quad \gamma &= \pi + k\pi \end{aligned}$$

with $k \in \mathbb{N}_+$.

Another signal is given as

$$\tilde{z}(t,\phi_1) = \Big|z(t,\phi_1) + 2j\sin(\omega(t)+\phi_1)\Big|.$$

(g) Determine the linear mean and variance of $\tilde{z}(t, \phi_1)$ for an arbitrary time instance $t = \beta$. (4 P)
$\tilde{z}(t, \phi_1)$ is identical 2 for all inputs. Thus, the mean and variance are given by $m_{\tilde{z}} = 2$ and $\sigma_{\tilde{z}}^2 = 0$, respectively.

(h) Is $\tilde{z}(t, \phi_1)$ ergodic? Give reason to your answer! (3 P)
Since $\tilde{z}(t, \phi_1) = 2$ holds, there is no time dependency, yielding stationarity. Since $\tilde{z}(t, \phi_1)$ is a constant, it is, additionally, irrelevant over which dimension an integration is performed. The combination of both attributes yields ergodicity for $\tilde{z}(t, \phi_1)$.

Part 3 *This part may be solved independently of Part 1 and 2.*

Given is the discrete noise process $v(n)$ with the constant power density S_0. An instance of the noise process is used as input for the transmission system $H(e^{j\Omega})$.

$$v(n) \longrightarrow \boxed{H(e^{j\Omega})} \longrightarrow a(n)$$

The autocorrelation function of the output $a(n)$ of the system was found to be:

$$s_{aa}(\kappa) = \begin{cases} -\frac{1}{2}|\kappa| + 1, & \text{for } |\kappa| \leq 2 \\ 0, & \text{else} \end{cases}$$

(j) Evaluate the autocorrelation function of $v(n)$ at $\kappa = 0, 1, 2$. (2 P)

$$s_{vv}(0) = S_0 \gamma_0(0)$$
$$s_{vv}(1) = s_{vv}(2) = 0$$

(k) Calculate $|H(e^{j\Omega})|$. Simplify your result! (5 P)

The autocorrelation function may be transformed to

$$S_{aa}(e^{j\Omega}) = 1 + \frac{1}{2}e^{j\Omega} + \frac{1}{2}e^{-j\Omega}$$
$$= 1 + \cos(\Omega)$$

Using $S_{aa}(e^{j\Omega}) = S_{vv}(e^{j\Omega})|H(e^{j\Omega})|^2$ the magnitude spectrum may be calculated:

$$|H(e^{j\Omega})| = \sqrt{\frac{S_{aa}(e^{j\Omega})}{S_{vv}(e^{j\Omega})}} = \sqrt{\frac{1 + \cos(\Omega)}{S_0}}$$

(l) Explain, using *(k)*, why signals like $v(n)$ are often used for system identification purposes. (2 P)

By exciting the system with a white noise signal, a division by zero is avoided when computing the transfer function. Additionally as there is no frequency dependency within the spectrum of the excitation signal, $|H(e^{j\Omega})|^2$ is then a scaled version of the output power density.

The same transmission system is now used for the input signal $i(n)$. The resulting autocorrelation function at the output of the system is measured as

$$s_{\tilde{a}\tilde{a}}(\kappa) = \begin{cases} 2 \cdot s_{aa}(\kappa), & for\ \kappa = 0 \\ s_{aa}(\kappa), & \text{else} \end{cases}$$

(m) Calculate $S_{ii}(e^{j\Omega})$. (6 P)

The autocorrelation function may be transformed to

$$S_{\tilde{a}\tilde{a}}(e^{j\Omega}) = 2 + \frac{1}{2}e^{j\Omega} + \frac{1}{2}e^{-j\Omega}$$
$$= 2 + \cos(\Omega)$$

Using $S_{\tilde{a}\tilde{a}}(e^{j\Omega}) = S_{ii}(e^{j\Omega})|H(e^{j\Omega})|^2$, the power density may be calculated:

$$S_{ii}(e^{j\Omega}) = \frac{S_{\tilde{a}\tilde{a}}(e^{j\Omega})}{|H(e^{j\Omega})|^2} = \frac{2+\cos(\Omega)}{1+\cos(\Omega)} S_0$$

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