Advanced Digital Signal Processing

Examination WS 2016/2017

Examiner: Prof. Dr.-Ing. Gerhard Schmidt

Date: 02.03.2017

Name: ______________________________

Matriculation Number: ______________________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorised to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ______________________________

Marking

Problem	1	2	3
Points	/35	/35	/30

Total number of points: __________________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated __________________ Signature: ______________________________

Advanced Digital Signal Processing

Examination WS 2016/2017

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	02.03.2017
Time:	09:00 h – 10:30 h (90 minutes)
Location:	KS2, C-SR I

Remarks

- Please check that you have received a cover sheet plus 4 sheets with 3 problems.
- Please write your **name** and your **matriculation number** on each sheet of paper that you return.
- Please keep your student ID and your identity card ready.
- During the exam only questions concerning the problems are answered.
- Please don't use any pencil or red pen.
- Please use a **new** sheet of paper with your name and matriculation number on it for **each problem**. You can ask for more sheets of paper, if necessary.
- The exam is open books, open notes; other people are closed. Programmable electronical devices except pocket calculators are not permitted.
- Partial credit will be given. No credit will be given if an answer appears with no supporting work or reason.
- Note that the given points of the subproblems are just preliminary.
- At the end of the exam put all sheets together as you have received them, including the problem sheets.
- No one is allowed to talk or to leave his or her seat until **all** exams have been collected.
- The problems and the solutions will be published on the website of the lecture. Also the date and the place of the inspection will be announced on this website.

Problem 1 (35 points)

This problem consists of three parts (a), (b) and (c). They are **not** related to each other and can be solved **independently**.

(a) Given are two continuous-time signals $x(t)$ and $y(t)$ with $f_1 = 1\,kHz$ and $f_2 = 2\,kHz$:

$$x(t) = 10 + 8\sin(2\cdot\pi\cdot f_1\cdot t) - 5\cos(2\cdot\pi\cdot f_2\cdot t)$$
$$y(t) = 8\cos(2\cdot\pi\cdot f_1\cdot t) + 10\cos(2\cdot\pi\cdot f_2\cdot t)$$

(i) Determine the Fourier transform $X(j\omega)$ and $Y(j\omega)$ for the signals $x(t)$ and $y(t)$, respectively. (6 P)

$x(t) = 10 + 8\cdot\sin(2000\pi s^{-1}\cdot t) - 5\cdot\cos(4000\pi s^{-1}\cdot t)$
$\Rightarrow X(j\omega) = 2\cdot\pi\cdot 10\cdot\delta_0(\omega) + 8\cdot\frac{\pi}{j}\cdot[\delta_0(\omega - \frac{2000\pi}{s}) - \delta_0(\omega + \frac{2000\pi}{s})]$
$-5\cdot\pi\cdot[\delta_0(\omega - \frac{4000\pi}{s}) + \delta_0(\omega + \frac{4000\pi}{s})]$
$\Rightarrow X(j\omega) = 2\cdot\pi\cdot[10\cdot\delta_0(\omega) + \frac{4}{j}\cdot[\delta_0(\omega - \frac{2000\pi}{s}) - \delta_0(\omega + \frac{2000\pi}{s})]$
$-\frac{5}{2}\cdot[\delta_0(\omega - \frac{4000\pi}{s}) + \delta_0(\omega + \frac{4000\pi}{s})]$ (3 P)

$y(t) = 8\cdot\cos(2000\pi s^{-1}\cdot t) + 10\cdot\cos(4000\pi s^{-1}\cdot t)$
$\Rightarrow Y(j\omega) = 8\cdot\pi\cdot[\delta_0(\omega - \frac{2000\pi}{s}) + \delta_0(\omega + \frac{2000\pi}{s})]$
$+10\cdot\pi\cdot[\delta_0(\omega - \frac{4000\pi}{s}) + \delta_0(\omega + \frac{4000\pi}{s})]$
$\Rightarrow Y(j\omega) = 2\cdot\pi\cdot[4\cdot[\delta_0(\omega - \frac{2000\pi}{s}) + \delta_0(\omega + \frac{2000\pi}{s})]$
$+5\cdot[\delta_0(\omega - \frac{4000\pi}{s}) + \delta_0(\omega + \frac{4000\pi}{s})]$ (3 P)

(ii) Both signals should be processed by one time-discrete system. What is the minimum sampling frequency f_s that has to be used? Give a reason. (2 P)

$f_{\max} = 2000$ Hz => $f_s = 4$ kHz (2 P)

(b) Now assume that the signal has been sampled with a sampling frequency of $f_s = 12$ kHz.

(i) Determine the time-discrete signals $x(n)$ and $y(n)$. (5 P)

$t = nT_s = \frac{n}{f_s} \Rightarrow x(\frac{n}{f_s}) \Rightarrow y(\frac{n}{f_s})$ (1 P)
$x(n) = 10 + 8\sin(\frac{\pi}{6}n) - 5\cos(\frac{\pi}{3}n)$ (2 P)
$y(n) = 8\cos(\frac{\pi}{6}n) + 10\cos(\frac{\pi}{3}n)$ (2 P)

(ii) The signal $y(n)$ is a result of $x(n)$ convolved with a system h. What kind of a filter is the system h (lowpass, bandpass or highpass)? Give reasons. (2 P)

Example solution: The system is a highpass filter, because the frequency component at $\Omega = 0$ is not passed through but frequencies above $\Omega = \pi/6$ are passed. (2 P)

(iii) Give values of the transfer function of $H(e^{j\Omega})$ at $\Omega = 0, \pi/6$ and $\pi/3$. (6 P)

$H(e^{j0}) = 0$ (2 P)
$H(e^{j\frac{\pi}{6}}) = e^{j\frac{\pi}{2}}$ (2 P)
$H(e^{j\frac{\pi}{3}}) = -2$ (2 P)

(c) Now consider an input signal $x(n) = \{1, 5, -2, -1, 6, 1, -2, 5, 1\}$ with $x(n) = 0$ for $n < 0, n > 8$ into a system $h(n) = \{1, 2, 1\}$

(i) Compute the corresponding output $y(n)$ of the system. (6 P)

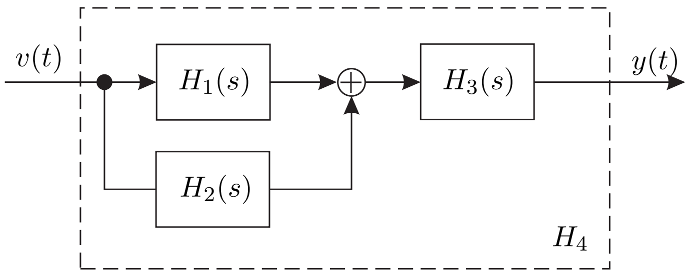
Each correct answer gives **0.5 P**
$y(0) = 1 \cdot 1 = \mathbf{1}$
$y(1) = (1 \cdot 2) + (1 \cdot 5) = \mathbf{7}$
$y(2) = (1 \cdot 1) + (5 \cdot 2) + (-2 \cdot 1) = \mathbf{9}$
$y(3) = (5 \cdot 1) + (-2 \cdot 2) + (-1 \cdot 1) = \mathbf{0}$
$y(4) = (-2 \cdot 1) + (-1 \cdot 2) + (6 \cdot 1) = \mathbf{2}$
$y(5) = (-1 \cdot 1) + (6 \cdot 2) + (1 \cdot 1) = \mathbf{12}$
$y(6) = (6 \cdot 1) + (1 \cdot 2) + (-2 \cdot 1) = \mathbf{6}$
$y(7) = (1 \cdot 1) + (-2 \cdot 2) + (5 \cdot 1) = \mathbf{2}$
$y(8) = (-2 \cdot 1) + (5 \cdot 2) + (1 \cdot 1) = \mathbf{9}$
$y(9) = (5 \cdot 1) + (1 \cdot 2) + (0 \cdot 1) = \mathbf{7}$
$y(10) = (1 \cdot 1) + (0 \cdot 2) + (0 \cdot 1) = \mathbf{1}$
$\Rightarrow \mathbf{y(n) = \{1,\ 7,\ 9,\ 0,\ 2,\ 12,\ 6,\ 2,\ 9,\ 7,\ 1\}}$

(ii) Compute the 8-point DFT $H(\mu)$ of $h(n)$. (8 P)
Each correct answer gives **1 P**

$H(\mu) = \sum_{n=0}^{7} h(n) \cdot W_8^{\mu n} = 1 + 2W_8^{\mu} + W_8^{2\mu}$
$H(0) = 4$
$H(1) = 1 + \sqrt{2} - j(1 + \sqrt{2})$
$H(2) = -2j$
$H(3) = 1 - \sqrt{2} + j(1 - \sqrt{2})$
$H(4) = 0$
$H(5) = H^*(3) = 1 - \sqrt{2} - j(1 - \sqrt{2})$
$H(6) = H^*(2) = 2j$
$H(7) = H^*(1) = 1 + \sqrt{2} + j(1 + \sqrt{2})$

Problem 2 (35 points)

Given is the following signal flow diagram:



The three systems can be described by the following transfer functions:

$$H_1(s) = \frac{-6s-6}{3s+6}$$
$$H_2(s) = \frac{3s-1}{s+2}$$
$$H_3(s) = \frac{s-2}{s+3}$$

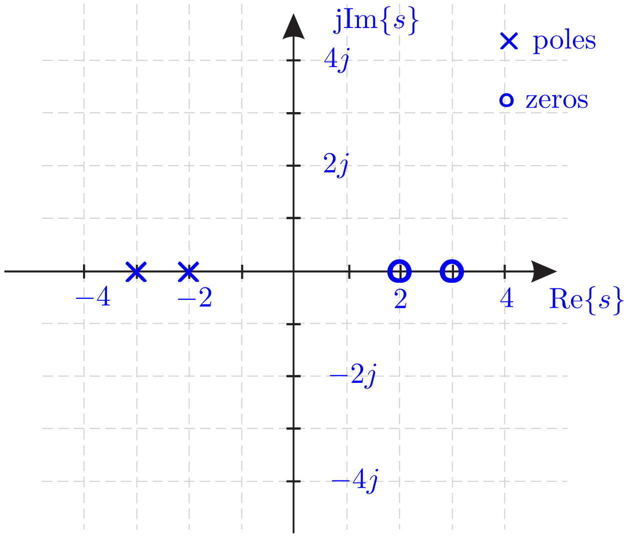
(a) Determine the transfer function of the total system $H_4(s)$. (5 P)

$$\begin{aligned} H_4(s) &= [H_1(s) + H_2(s)] \cdot H_3(s) \\ &= \left[\frac{-6s-6}{3s+6} + \frac{3s-1}{s+2}\right] \cdot \frac{s-2}{s+3} \\ &= \left[\frac{-2s-2}{s+2} + \frac{3s-1}{s+2}\right] \cdot \frac{s-2}{s+3} \\ &= \frac{s-3}{s+2} \cdot \frac{s-2}{s+3} \\ &= \frac{(s-3)(s-2)}{(s+2)(s+3)} \\ &= \frac{s^2-5s+6}{s^2+5s+6} \end{aligned}$$

(b) Calculate the poles and zeros of the system H_4 and sketch the pole-zero diagram. (4 P)

Determine poles and zeros directly from transfer function:

$$s_{0,1} = 3, \quad s_{0,2} = 2, \quad s_{\infty,1} = -3, \quad s_{\infty,2} = -2$$



(c) Which type of filter is described by $H_4(s)$, i.e. low-, high-, bandpass etc.? Give reason to your answer. (2 P)

The system is an allpass-filter since $s_{0,i} = -s^*_{\infty,i}$ holds for all poles and zeros.

(d) Determine the impulse response $h_{0,4}(t)$ of the filter described by $H_4(s)$. (6 P)

Polynomal long division:

$$\begin{array}{l} \left(\quad s^2 \quad - 5s + 6\right) : \left(s^2 + 5s + 6\right) = 1 + \dfrac{-10s}{s^2 + 5s + 6} \\ \underline{\;- s^2 \quad - 5s - 6\;} \\ \qquad\quad - 10s \end{array}$$

Partial fraction expansion:

$$\begin{aligned} H_4(s) &= \frac{A_1}{s+3} + \frac{A_2}{s+2} + 1 \\ A_1 &= \frac{-10s}{(s+3)(s+2)}(s+3)\bigg|_{s=-3} = \frac{-10s}{s+2}\bigg|_{s=-3} = -30 \\ A_2 &= \frac{-10s}{(s+3)(s+2)}(s+2)\bigg|_{s=-2} = \frac{-10s}{s+3}\bigg|_{s=-2} = 20 \\ \Rightarrow H_4(s) &= \frac{-30}{s+3} + \frac{20}{s+2} + 1 \end{aligned}$$

Inverse Laplace transformation:

$$\begin{aligned} h_{0,4}(t) &= \left(\sum_{i=1}^{2} A_i \cdot \mathrm{e}^{s_{\infty,i}t}\right) \cdot \delta_{-1}(t) + \delta_0(t) \\ &= \left(-30\mathrm{e}^{-3t} + 20\mathrm{e}^{-2t}\right) \cdot \delta_{-1}(t) + \delta_0(t) \end{aligned}$$

An 'equivalent' digital filter has to be found by the so called *impulse-invariance* method. Its sampling period is defined as T_s.

(e) Give the corresponding discrete impulse response $h_{4,1}(n)$ in general and especially for the case $T_s = 1$. (3 P)

$h_{4,1}(n) = h_{0,4}(t = n \cdot T_s) = \left(-30\mathrm{e}^{-3T_s n} + 20\mathrm{e}^{-2T_s n}\right)\gamma_{-1}(n) + \gamma_0(n)$
$T_s = 1$: $h_{4,1}(n) = \left(-30\mathrm{e}^{-3n} + 20\mathrm{e}^{-2n}\right)\gamma_{-1}(n) + \gamma_0(n)$

(f) Determine the corresponding digital filter transfer function $H_{4,1}(z)$ for the case $T_s = 1$. (3 P)

Z-transform of $h_{4,1}(n)$:

$$
\begin{aligned}
H(z) &= \sum_{i=1}^{2} \frac{A_i}{1 - \mathrm{e}^{-s_{\infty,i} T_s} z^{-1}} + 1 \\
&= \frac{-30}{1 - \mathrm{e}^{-3} z^{-1}} + \frac{20}{1 - \mathrm{e}^{-2} z^{-1}} + 1
\end{aligned}
$$

As an alternative, a digital filter can be found via the so-called *bilinear transformation*.

(g) Determine the transfer function $H_{4,2}(z)$ using the bilinear transformation with $T_s = 1$. (6 P)

$$
\begin{aligned}
H_{4,2}(z) &= H\left(\frac{2}{T_s}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right) \\
&= H\left(2\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right) \\
&= \frac{-30}{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 3} + \frac{20}{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 2} + 1 \\
&= \frac{-30(1 + z^{-1})}{2(1 - z^{-1}) + 3(1 + z^{-1})} + \frac{20(1 + z^{-1})}{2(1 - z^{-1}) + 2(1 + z^{-1})} + 1 \\
&= \frac{-30 - 30z^{-1}}{z^{-1} + 5} + 5(1 + z^{-1}) + 1
\end{aligned}
$$

(h) Which signal components (i.e. impulses, unit step functions, exponential functions etc.) are included within the impulse response $h_{4,2}(n)$ corresponding to the transfer function $H_{4,2}(z)$? Explain your answer. (3 P)

The impulse response $h_{4,2}(n)$ contains weighted impulses and weighted unit-step functions due to the given form of $H_{4,2}(z)$. $H_{4,2}(z)$ can be written in terms of geometric series. The inverse $\mathcal{Z}$-transform lead to impulses and exponential weighted unit-step sequences.

(i) Compare your result of part (h) to $h_{4,1}(n)$: Are both impulse responses identical? Explain your observation. (3 P)

The functions are not identical because the impulse response $h_{4,2}(n)$ is <u>not</u> unique.

Problem 3 (30 points)

This problem consists of two parts (a) and (b). They are **not** related to each other and can be solved **independently**.

(a) Given are the input $X(e^{j\Omega})$ and the output spectra $Y(e^{j\Omega})$ of a multi-rate system:

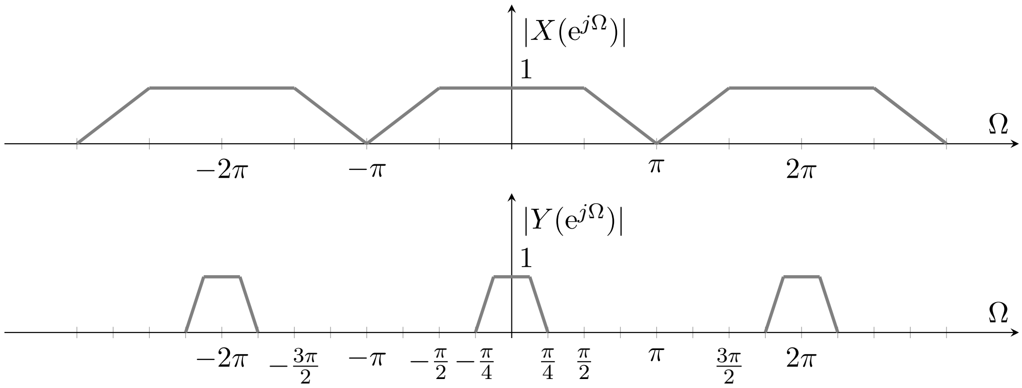


Figure 1: Input and output spectra of a multi-rate system.

(i) Sketch a block diagram of a system which produces output spectrum $Y(e^{j\Omega})$. If you use any filters, please plot the frequency response (magnitude is sufficient) of those filters. (11 P)

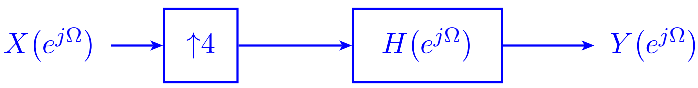


Figure 2: Block diagram of a multi-rate system.

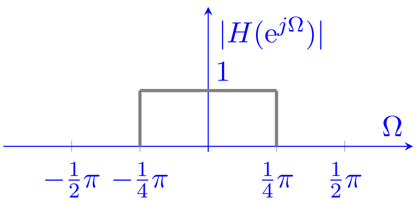


Figure 3: Frequency responses of the system function.

(ii) What is the purpose of this system? (2 P)

interpolation

(iii) The signal with the spectrum $X(e^{j\Omega})$ should be downsampled by a factor M = 3. Sketch the ideal low pass filter $H(e^{j\Omega})$ such that no aliasing occurs. Mark all axes of your plot. (5 P)

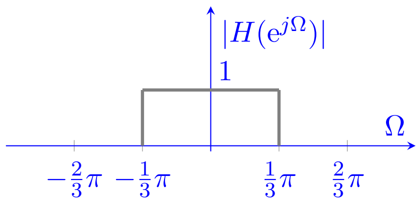


Figure 4: Frequency responses of the system function.

(b) Consider the block diagram of a multi-rate system as depicted in Fig. 5.

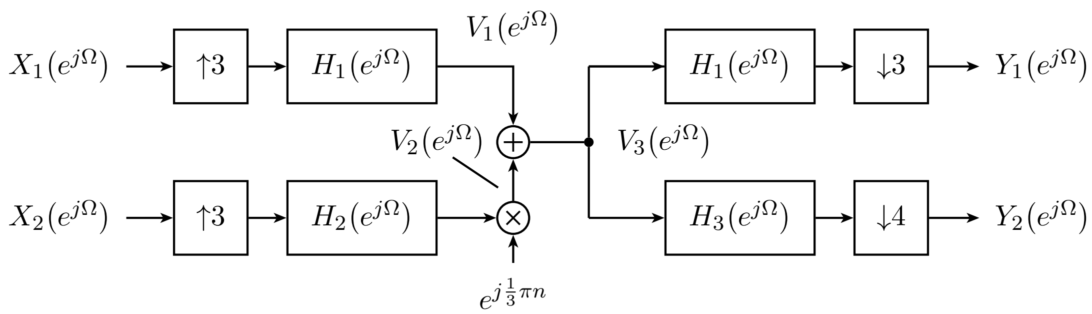


Figure 5: Block diagram of a multi-rate system.

(i) Give expressions for $V_1(e^{j\Omega})$, $V_2(e^{j\Omega})$ and $V_3(e^{j\Omega})$ in terms of $X_1(e^{j\Omega})$ and $X_2(e^{j\Omega})$. (6 P)

$$V_1(e^{j\Omega}) = X_1(e^{j\Omega 3})H_1(e^{j\Omega}).$$
$$V_2(e^{j\Omega}) = X_2(e^{j\Omega 3-\frac{1}{3}\pi})H_2(e^{j\Omega-\frac{1}{3}\pi}).$$
$$V_3(e^{j\Omega}) = X_1(e^{j\Omega 3})H_1(e^{j\Omega}) + X_2(e^{j\Omega 3-\frac{1}{3}\pi})H_2(e^{j\Omega-\frac{1}{3}\pi}).$$

(ii) Give expressions for $Y_1(e^{j\Omega})$ and $Y_2(e^{j\Omega})$ in terms of $V_3(e^{j\Omega})$. (6 P)

$$\begin{aligned}
Y_1(e^{j\Omega}) =& \frac{1}{M}\sum_{k=0}^{M-1} V_3\left(\mathrm{e}^{j\frac{\Omega-2\pi k}{M}}\right) H_1\left(\mathrm{e}^{j\frac{\Omega-2\pi k}{M}}\right)\\
=& \frac{1}{3}\left[V_3\left(\mathrm{e}^{j\frac{\Omega}{3}}\right) H_1\left(\mathrm{e}^{j\frac{\Omega}{3}}\right) + \dots\right.\\
& V_3\left(\mathrm{e}^{j\frac{\Omega-2\pi}{3}}\right) H_1\left(\mathrm{e}^{j\frac{\Omega-2\pi}{3}}\right) + \dots\\
& \left.V_3\left(\mathrm{e}^{j\frac{\Omega-4\pi}{3}}\right) H_1\left(\mathrm{e}^{j\frac{\Omega-4\pi}{3}}\right)\right].
\end{aligned}$$

$$\begin{aligned}
Y_2(e^{j\Omega}) =& \frac{1}{M}\sum_{k=0}^{M-1} V_3\left(\mathrm{e}^{j\frac{\Omega-2\pi k}{M}}\right) H_3\left(\mathrm{e}^{j\frac{\Omega-2\pi k}{M}}\right)\\
=& \frac{1}{4}\left[V_3\left(\mathrm{e}^{j\frac{\Omega}{4}}\right) H_3\left(\mathrm{e}^{j\frac{\Omega}{4}}\right) + \dots\right.\\
& V_3\left(\mathrm{e}^{j\frac{\Omega-2\pi}{4}}\right) H_3\left(\mathrm{e}^{j\frac{\Omega-2\pi}{4}}\right) + \dots\\
& V_3\left(\mathrm{e}^{j\frac{\Omega-4\pi}{4}}\right) H_3\left(\mathrm{e}^{j\frac{\Omega-4\pi}{4}}\right) + \dots\\
& \left.V_3\left(\mathrm{e}^{j\frac{\Omega-6\pi}{4}}\right) H_3\left(\mathrm{e}^{j\frac{\Omega-6\pi}{4}}\right)\right].
\end{aligned}$$