FACULTY OF ENGINEERING
CHRISTIAN-ALBRECHTS-UNIVERSITÄT
ZU KIEL

DIGITAL
SIGNAL PROCESSING AND
SYSTEM THEORY

Advanced Digital Signal Processing

Examination WS 2015

Examiner: Prof. Dr.-Ing. Gerhard Schmidt

Date: 07.03.2016

Name: ______________________________

Matriculation Number: ______________________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorised to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ______________________________

Marking

Problem	1	2	3
Points	/27	/40	/33

Total number of points: ________________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated ________________ Signature: ______________________________

Advanced Digital Signal Processing

Examination WS 2015

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	07.03.2016
Time:	09:00 h – 10:30 h (90 minutes)
Location:	KS2, C-SR I

Remarks

- Please check that you have received a cover sheet plus 3 sheets with 3 problems.
- Please write your **name** and your **matriculation number** on each sheet of paper that you return.
- Please keep your student ID and your identity card ready.
- During the exam only questions concerning the problems are answered.
- Please don't use any pencil or red pen.
- Please use a **new** sheet of paper with your name and matriculation number on it for **each problem**. You can ask for more sheets of paper, if necessary.
- The exam is open books, open notes; other people are closed. Programmable electronical devices except pocket calculators are not permitted.
- Partial credit will be given. No credit will be given if an answer appears with no supporting work or reason.
- Note that the given points of the subproblems are just preliminary.
- At the end of the exam put all sheets together as you have received them, including the problem sheets.
- No one is allowed to talk or to leave his or her seat until **all** exams have been collected.
- The problems and the solutions will be published on the website of the lecture. Also the date and the place of the inspection will be announced on this website.

Problem 1 (27 points)

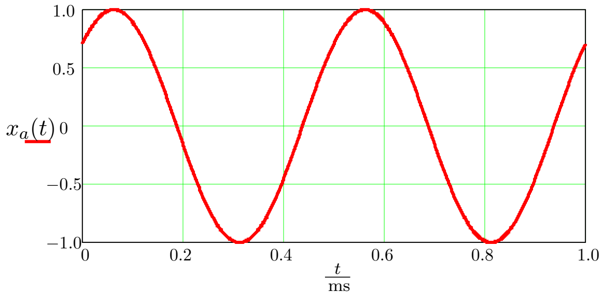
This question consists of two parts (a) and (b). They are **not** related to each other and can be solved independently.

(a) Given is the following continuous-time signal $x_a(t)$:

$$x_a(t) = \sin(2 \cdot \pi \cdot 2\,\text{kHz} \cdot t + \frac{\pi}{4})$$

The discrete time signal $x(n)$ is generated by sampling $x_a(t)$ with the sampling rate $f_s = 5000$ Hz.

(i) Sketch the continuous-time signal $x_a(t)$ in the time interval from 0 to 1 ms. (3 P)

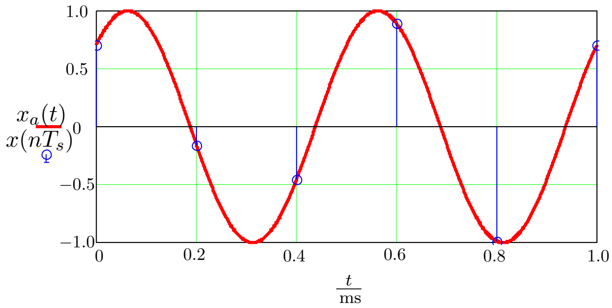


(ii) Calculate the sampled version $x(n)$ of the continuous-time signal $x_a(t)$ in the general form. Simplify the result as much as possible. (2 P)

$$x(n) = x_a(n \cdot T_s) = \sin(2 \cdot \pi \cdot \frac{2\,\text{kHz}}{5\,\text{kHz}} \cdot n + \frac{\pi}{4}) = \sin(\pi \cdot \frac{4}{5} \cdot n + \frac{\pi}{4})$$

(iii) Calculate $x(n)$ for $0 \leqslant n \leqslant 5$ and round the values to four decimal places. (3 P)
$x(0) \approx 0.7071$; $x(1) \approx -0.1564$; $x(2) \approx -0.4540$; (1.5 P)
$x(3) \approx 0.8910$; $x(4) \approx -0.9877$; $x(5) \approx 0.7071$ (1.5 P)

(iv) Sketch $x(n)$ in the drawing of part (a,i). (2 P)



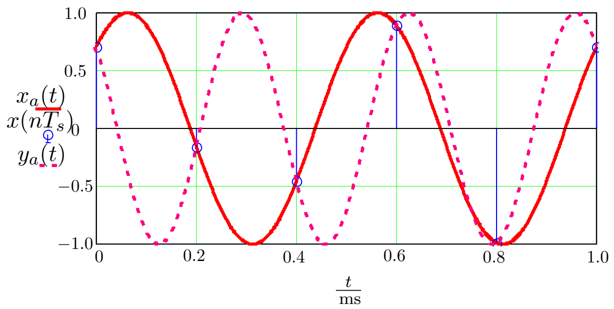
(v) Calculate the frequency f_{alias} and the phase ϕ_{alias} of the continues-time signal $y_a(t) = \sin(2 \cdot \pi \cdot f_{alias} \cdot t + \phi_{alias})$ under the condition $y(n) = x(n)$. **Hint:** (3 P)

The frequency f_{alias} has to be between $0.5 \cdot f_s$ and f_s and also the condition $y(n) = y_a(\frac{n}{f_s})$ has to be fulfilled.

In general the following occurs when the sampled signal has a sinusoidal component that is greater than $f_s/2$, but less than f_s. The frequency f can be replaced with $(f_s - f_0)$. It follows:
$x(n) = \sin(2 \cdot \pi \cdot f \cdot n \cdot T_s + \phi)$
$x(n) = \sin(2 \cdot \pi \cdot (f_s - f_0) \cdot n \cdot T_s + \phi)$
$x(n) = \sin(2 \cdot \pi \cdot f_s \cdot T_s - 2 \cdot \pi \cdot f_0 \cdot n \cdot T_s + \phi)$
$x(n) = \sin(2 \cdot \pi \cdot n - 2 \cdot \pi \cdot f_0 \cdot n \cdot T_s + \phi)$
$x(n) = \sin(-(2 \cdot \pi \cdot f_0 \cdot n \cdot T_s - \phi))$
$x(n) = \sin(2 \cdot \pi \cdot f_0 \cdot n \cdot T_s - \phi \pm \pi)$

$f_{alias} = f_s - f_0 = 5\,\text{kHz} - 2\,\text{kHz} = 3\,\text{kHz}$ (2 P)
$\phi_{alias} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ or $\phi_{alias} = -\pi - \frac{\pi}{4} = \frac{5\pi}{4}$ (1 P)

(vi) Sketch $y_a(t)$ also in the drawing of part (a,i). (3 P)



(b) Given is the 4-point DFT of a filter impulse response $h(n)$ as

$$H(\mu) = \{2, -j, 0, j\}$$

and an input signal $x(n) = \{1, 1, -1, 5\}$.

(i) Find the 4-point DFT $X(\mu)$ for $x(n)$ for $\mu = 0 \ldots 3$. (5 P)

$X(0) = 1 + 1 - 1 + 5 = \mathbf{6}$ (1 P)
$X(1) = 1 - j + 1 + 5j = \mathbf{2{+}4j}$ (1 P)
$X(2) = 1 - 1 - 1 - 5 = \mathbf{-6}$ (1 P)
$X(3) = 1 + j + 1 - 5j = \mathbf{2{-}4j}$ (1 P)
$X(\mu) = \{6, 2 + 4j, -6, 2 - 4j\}$ (1 P)

(ii) Find the output signal $y(n) = x(n) \circledast h(n)$ by first finding $Y(\mu) = \text{DFT}\{y(n)\}$. (6 P)

$Y(\mu) = X(\mu) \cdot H(\mu)$
$Y(\mu) = \{6, 2 + 4j, -6, 2 - 4j\} \cdot \{2, -j, 0, j\}$
$Y(\mu) = \{12, 4 - 2j, 0, 4 + 2j\}$ (2 P)

$y(0) = \frac{1}{4} \cdot (12 + 4 - 2j + 4 + 2j) = \mathbf{5}$ (1 P)

$y(1) = \frac{1}{4} \cdot (12 + 2 + 4j + 2 - 4j) = \mathbf{4}$ (1 P)

$y(2) = \frac{1}{4} \cdot (12 - 4 + 2j - 4 - 2j) = \mathbf{1}$ (1 P)

$y(3) = \frac{1}{4} \cdot (12 - 2 - 4j - 2 + 4j) = \mathbf{2}$ (1 P)

$y(n) = \{5, 4, 1, 2\}$

Problem 2 (40 points)

This question consists of three parts a, b and c. They are **not** related to each other and can be solved independently.

(a) A digital lowpass filter is required to meet the following specifications:

- Passband ripple: $\leq 0.5\,\text{dB}$
- Passband edge: $2.5\,\text{kHz}$
- Stopband attenuation: $\geq 50\,\text{dB}$
- Stopband edge: $3.0\,\text{kHz}$
- Sample rate: $16\,\text{kHz}$

The filter is to be designed by using the impulse invariance method on an analog system function.

(i) Determine what approximate order Butterworth and Chebyshev analog design must be used to meet the specifications in the digital implementation. (8 P)

First we determine the values for δ_1 and δ_2. They are used in the magnitude representation of filter specifications.

$$\begin{aligned} \delta_1 &= 1 - 10^{-0.5/20} = 0.055939123 \\ \delta_2 &= 10^{-50/20} = 0.00316227766 \end{aligned}$$

δ_1 is related to ϵ by

$$\begin{aligned} (1-\delta_1)^2 &= \frac{1}{1+\epsilon^2} \\ \epsilon^2 &= \frac{1}{(1-\delta_1)^2} - 1 \\ \epsilon &= 0.3493113978 \end{aligned}$$

and δ is given by

$$\delta = \sqrt{\frac{1}{\delta_2^2} - 1} = 316.2261849$$

The normalized frequencies for the passband edge and stopband edge in the digital domain are given by

$$\begin{aligned} \Omega_p &= 2\pi \cdot (2.5 \times 10^3)/(16 \times 10^3) \\ \Omega_s &= 2\pi \cdot (3 \times 10^3)/(16 \times 10^3) \end{aligned}$$

We determine the passband edge and stopband edge in the analog domain by inverse transformation of the values for the digital domain:

$$\begin{aligned} \omega_p &= 1/T \cdot \Omega_p \\ \omega_s &= 1/T \cdot \Omega_s \\ \omega_s/\omega_p &= 1.2 \end{aligned}$$

This leads to the following filter length:

$$\begin{aligned}\text{Butterworth filter: } N_{min} &\geq \frac{\log_{10}(\delta/\epsilon)}{\log_{10}(\omega_s/\omega_p)} = 37.34194663 \Rightarrow 38\\ \text{Chebyshev filter: } N_{min} &\geq \frac{\log_{10}((\sqrt{1-\delta_2^2}+\sqrt{1-\delta_2^2(1+\epsilon^2)})/(\epsilon\delta_2))}{\log_{10}(\omega_s/\omega_p+\sqrt{(\omega_s/\omega_p)^2-1})}\\ &\geq 12.0531 \Rightarrow 13\end{aligned}$$

(ii) Which linear-phase filter types can be used to realize such a filter? (2 P)

As the desired filter has lowpass characteristic, only linear-phase filter types I an II can be used.

(iii) Sketch the design scheme of the lowpass filter. Mark the digital-filter passband and stopband edges, the passband, the transition band, the stopband, the passband ripple and the stopband attenuation. (5 P)

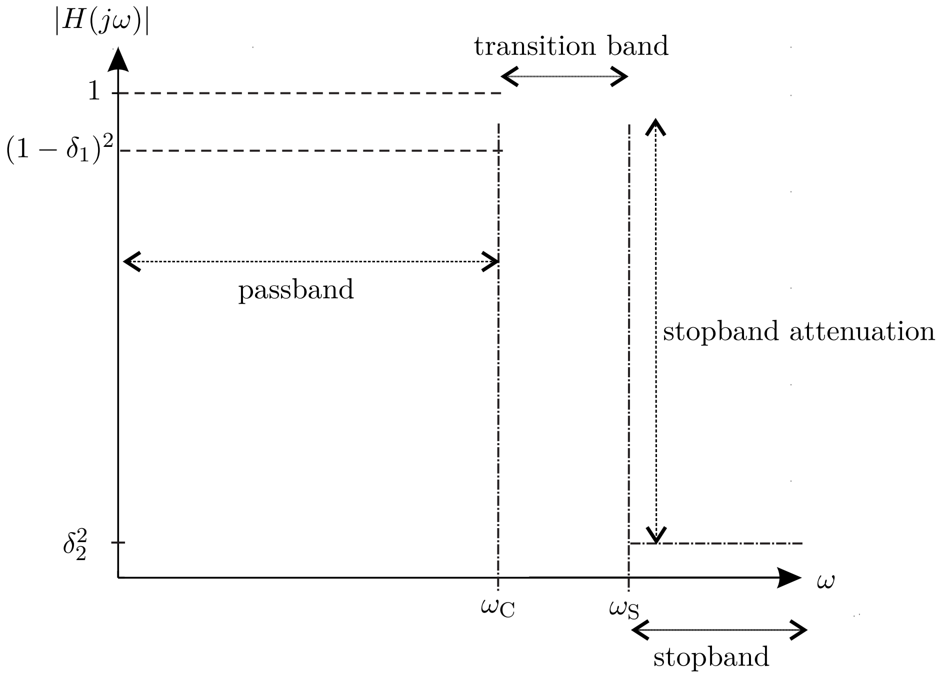


Figure 1: Design scheme of lowpass filter $|H(j\omega)|$.

(iv) Give an example (sketch) of a type-I-Chebyshev filter that will meet the specification. Include the example in the sketch from problem part (a,iii). (2 P)

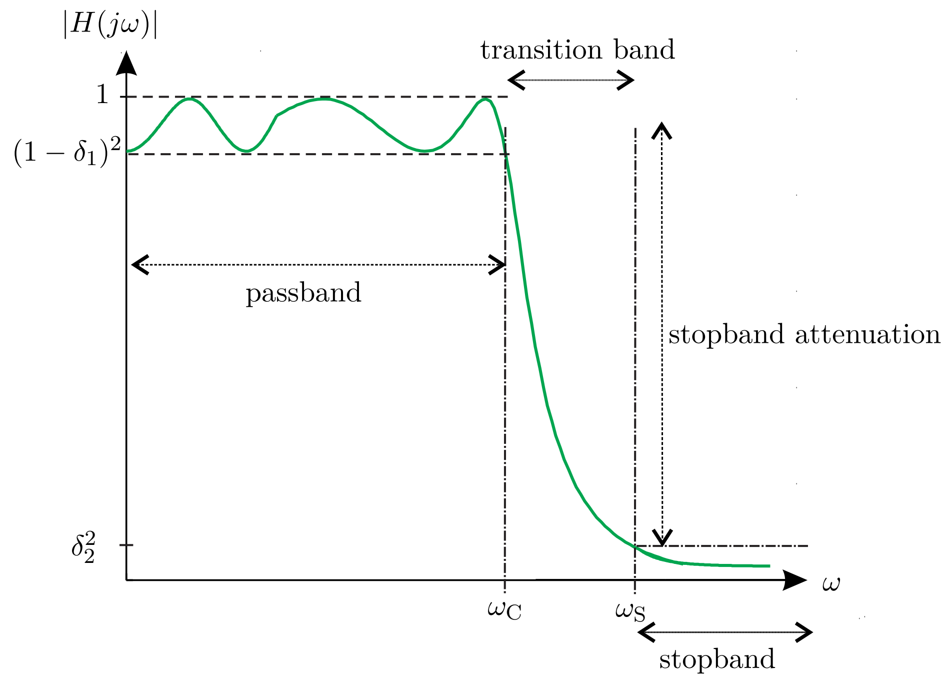


Figure 2: Design scheme of lowpass filter $|H(j\omega)|$.

(b) The transfer function $H(s)$ of a time continuous system is described by

$$H(s) = \frac{s^2+7s+12}{s^2+6s+8}$$

(i) Is the given system $H(s)$ stable? Give reason to your answer. (3 P)

For stability all Poles have to be in the left s-plane, which means $s_{\infty,i} < 0 \quad \forall i$. The poles can be determined by setting the denominator of the transfer function $H(s)$ to zero, which leads to:

$$s^2 + 6s + 8 = 0$$

To determine the poles, the p-q-formula is used:

$$\begin{aligned} s_{\infty,1/2} &= -3 \pm \sqrt{9-8} \\ &= -3 \pm \sqrt{1} \end{aligned}$$

This means the filter is stable, because both poles $s_{\infty,1} = -2$ and $s_{\infty,2} = -4$ are lying on the left s-plane.

(ii) Determine the corresponding $\mathcal{Z}$-domain transfer function $H(z)$ by using the bilinear transform for a sampling period $T = \frac{1}{2}$. (9 P)

Rearranging H(s) leads to

$$H(s) = \frac{(s+3)(s+4)}{(s+2)(s+4)}.$$

Canceling the Zero with the Pole at -4 leads to

$$H(s) = \frac{(s+3)}{(s+2)}.$$

Substitution of s by

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right).$$

We get for the $\mathcal{Z}$-Domain Transfer function $H(z)$:

$$\begin{aligned} H(z) &= \frac{\left(4\frac{1-z^{-1}}{1+z^{-1}}\right)+3}{\left(4\frac{1-z^{-1}}{1+z^{-1}}\right)+2} \\ &= \frac{4(1-z^{-1})+3(1+z^{-1})}{4(1-z^{-1})+2(1+z^{-1})} \\ &= \frac{4-4z^{-1}+3+3z^{-1}}{4-4z^{-1}+2+2z^{-1}} \\ &= \frac{7-z^{-1}}{6-2z^{-1}} \\ &= \frac{7z-1}{6z-2} \end{aligned}$$

(c) Consider a discrete filter $H_1(z)$ given by the following transfer function

$$H_1(z) = \frac{\frac{3}{2}z^2-\frac{1}{2}}{z^3-\frac{3}{2}z^2+\frac{1}{2}z}$$

(i) Divide the system $H_1(z)$ into a sum of real-valued first and second order sub-systems, such as

$$H_1(z) = \frac{b_{10}z^{-1}}{1+a_{10}z^{-1}} + \frac{b_{11}z^{-1}+b_{21}z^{-2}}{1+a_{11}z^{-1}}.$$

(7 P)

First determination of the poles:

$$z^3 - \frac{3}{2}z^2 + \frac{1}{2}z = 0$$

The first pole is obvious:

$$z_{\infty,1} = 0$$

Determining the remaining poles:

$$z^2 - \frac{3}{2}z + \frac{1}{2} = 0$$

$$
\begin{aligned}
z_{\infty,2/3} &= \frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{2}} \\
&= \frac{3}{4} \pm \sqrt{\frac{1}{16}} \\
&= \frac{3}{4} \pm \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
z_{\infty,2} &= 1 \\
z_{\infty,3} &= \frac{1}{2}
\end{aligned}
$$

Split the system using partial fraction expansion:

$$
\begin{aligned}
H_1(z) &= \frac{\frac{3}{2}z^2 - \frac{1}{2}}{z^3 - \frac{3}{2}z^2 + \frac{1}{2}z} \\
&= \frac{A}{z} + \frac{B}{(z-1)} + \frac{C}{(z-\frac{1}{2})}
\end{aligned}
$$

Determination of A, B and C:

$$
\begin{aligned}
A &= \frac{\frac{3}{2}z^2 - \frac{1}{2}}{(z-1)(z-\frac{1}{2})}\Big|_{z=0} \\
&= \frac{-\frac{1}{2}}{\frac{1}{2}} = -1 \\
B &= \frac{\frac{3}{2}z^2 - \frac{1}{2}}{z(z-\frac{1}{2})}\Big|_{z=1} \\
&= \frac{1}{\frac{1}{2}} = 2 \\
C &= \frac{\frac{3}{2}z^2 - \frac{1}{2}}{z(z-1)}\Big|_{z=\frac{1}{2}} \\
&= \frac{-\frac{1}{8}}{-\frac{1}{4}} = \frac{1}{2}
\end{aligned}
$$

Determination of $H_1(z)$ divided into real valued Sub-Systems:

$$\begin{aligned}
H_1(z) &= \frac{\frac{3}{2}z^2 - \frac{1}{2}}{z^3 - \frac{3}{2}z^2 + \frac{1}{2}z} \\
&= \frac{A}{z} + \frac{B}{(z-1)} + \frac{C}{(z-\frac{1}{2})} \\
&= -\frac{1}{z} + \frac{2}{(z-1)} + \frac{\frac{1}{2}}{(z-\frac{1}{2})} \\
&= \frac{-(z-1)+2z}{z(z-1)} + \frac{\frac{1}{2}}{(z-\frac{1}{2})} \\
&= \frac{z+1}{z^2-z} + \frac{\frac{1}{2}}{z-\frac{1}{2}} \\
&= \frac{z^{-1}+z^{-2}}{1-z^{-1}} + \frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}
\end{aligned}$$

(ii) Sketch the parallel form realization of the system using the subsystems from problem part (c,i) in direct form II. (4 P)

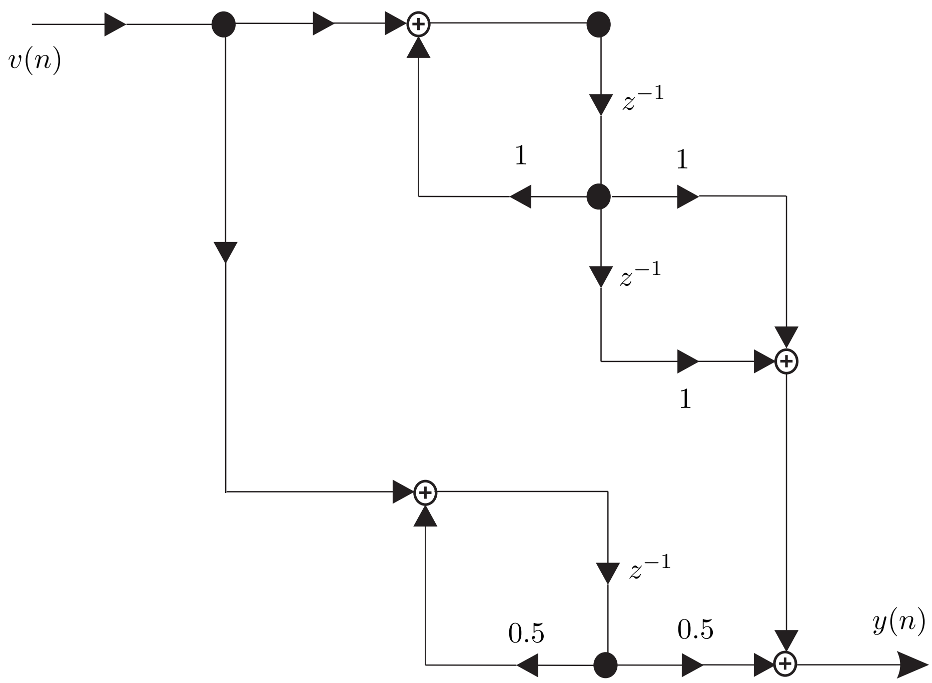


Figure 3: Signal flow graph of $H_1(z)$ in parallel form.

Problem 3 (33 points)

The two parts *(a)*, and *(b)* can be solved independently.

(a) Consider the multi-rate system depicted in Figure 1 with the system functions $H_1(e^{j\Omega})$, $H_3(e^{j\Omega})$, and $H_4(e^{j\Omega})$ as given in Figure 2 and the input spectra $X_1(e^{j\Omega})$ and $X_2(e^{j\Omega})$ as depicted in Figure 3. The system function $H_2(e^{j\Omega})$ is **not** known.

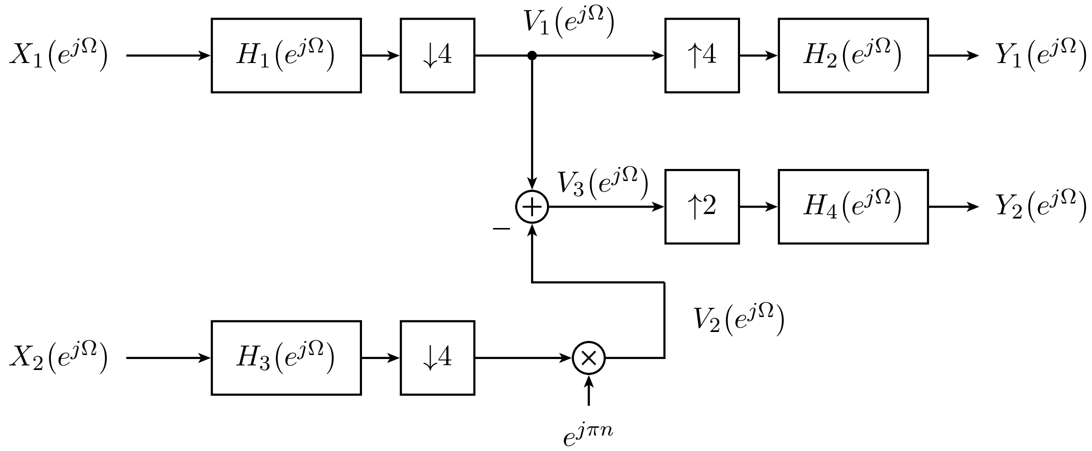


Figure 4: Block diagram of a multi-rate system

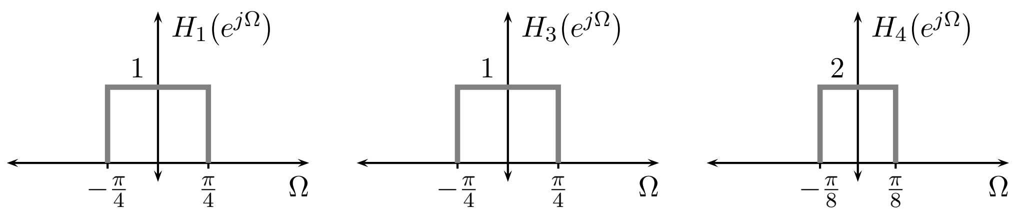


Figure 5: Frequency responses of the system functions

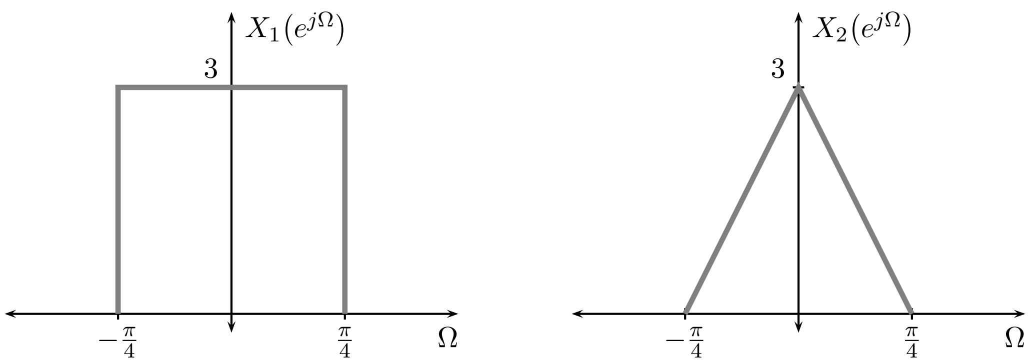
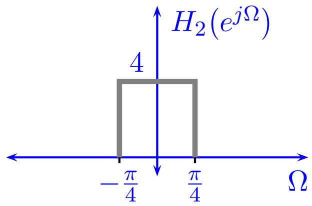


Figure 6: Frequency responses of the input signals

(i) Design a filter $H_2(e^{j\Omega})$ such that the output signal $Y_1(e^{j\Omega}) = X_1(e^{j\Omega})$. Sketch the corresponding frequency response of the filter. (3 P)



(ii) Give expressions for $V_1(e^{j\Omega})$, $Y_1(e^{j\Omega})$ and $V_2(e^{j\Omega})$ in terms of $X_1(e^{j\Omega})$, $X_2(e^{j\Omega})$, $H_1(e^{j\Omega})$, $H_2(e^{j\Omega})$, and $H_3(e^{j\Omega})$. Do **not** assume that $H_2(e^{j\Omega})$ is designed as in part *(i)*.

Additionally give a general expression for $Y_2(e^{j\Omega})$ in terms of $V_1(e^{j\Omega})$, $V_2(e^{j\Omega})$, and $H_4(e^{j\Omega})$. (7 P)

Intermediate spectrum $V_1(e^{j\Omega})$:

$$\begin{aligned} V_1(e^{j\Omega}) &= \frac{1}{M}\sum_{n=0}^{M-1} X_1\left(e^{j(\Omega-2\pi n)/M}\right) H_1\left(e^{j(\Omega-2\pi n)/M}\right) \\ &= \frac{1}{4}\Big[X_1\left(e^{j\Omega/4}\right) H_1\left(e^{j\Omega/4}\right) + X_1\left(e^{j(\Omega-2\pi)/4}\right) H_1\left(e^{j(\Omega-2\pi)/4}\right) + \dots \\ &\quad X_1\left(e^{j(\Omega-4\pi)/4}\right) H_1\left(e^{j(\Omega-4\pi)/4}\right) + X_1\left(e^{j(\Omega-6\pi)/4}\right) H_1\left(e^{j(\Omega-6\pi)/4}\right)\Big] \end{aligned}$$

(2 P)

Output spectrum $Y_1(e^{j\Omega})$:

$$\begin{aligned} Y_1(e^{j\Omega}) &= V_1\left(e^{j4\Omega}\right) H_2(e^{j\Omega}) \\ &= \frac{1}{4}\Big[X_1\left(e^{j4\Omega/4}\right) H_1\left(e^{j4\Omega/4}\right) + X_1\left(e^{j(4\Omega-2\pi)/4}\right) H_1\left(e^{j(4\Omega-2\pi)/4}\right) + \dots \\ &\quad X_1\left(e^{j(4\Omega-4\pi)/4}\right) H_1\left(e^{j(4\Omega-4\pi)/4}\right) + \dots \\ &\quad X_1\left(e^{j(4\Omega-6\pi)/4}\right) H_1\left(e^{j(4\Omega-6\pi)/4}\right)\Big] H_2(e^{j\Omega}) \end{aligned}$$

(2 P)

Intermediate spectrum $V_2(e^{j\Omega})$:

$$\begin{aligned} V_2(e^{j\Omega}) &= \left[\frac{1}{M}\sum_{n=0}^{M-1} X_2\left(e^{j\frac{\Omega-2\pi n}{M}-\pi}\right) H_3\left(e^{\frac{\Omega-2\pi n}{M}-\pi}\right)\right] \\ &= \frac{1}{4}\Big[X_2\left(e^{j\frac{\Omega}{4}-\pi}\right) H_3\left(e^{j\frac{\Omega}{4}-\pi}\right) + \dots \\ &\quad X_2\left(e^{j\frac{\Omega-2\pi}{4}-\pi}\right) H_3\left(e^{j\frac{\Omega-2\pi}{4}-\pi}\right) + \dots \\ &\quad X_2\left(e^{j\frac{\Omega-4\pi}{4}-\pi}\right) H_3\left(e^{j\frac{\Omega-4\pi}{4}-\pi}\right) + \dots \\ &\quad X_2\left(e^{j\frac{\Omega-6\pi}{4}-\pi}\right) H_3\left(e^{j\frac{\Omega-6\pi}{4}-\pi}\right)\Big] \end{aligned}$$

(2 P)

General expression for $Y_2(e^{j\Omega})$:

$$Y_2(e^{j\Omega}) = \left[V_1\left(e^{j2\Omega}\right) - V_2\left(e^{j2\Omega}\right)\right] H_4(e^{j\Omega})$$

(1 P)

(iii) Sketch $V_1(e^{j\Omega})$, $V_2(e^{j\Omega})$, $V_3(e^{j\Omega})$, and $Y_2(e^{j\Omega})$. (8 P)

The spectra are depicted in Figures **??**-**??**.

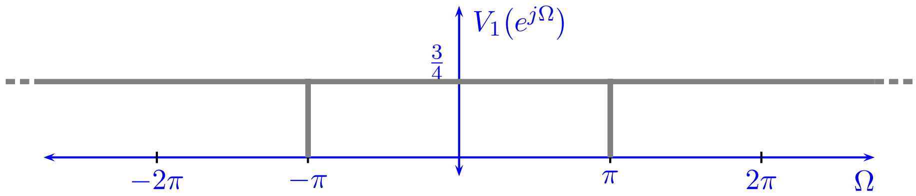


Figure 7: Frequency response of the intermediate signal $V_1(e^{j\Omega})$

(2 P)

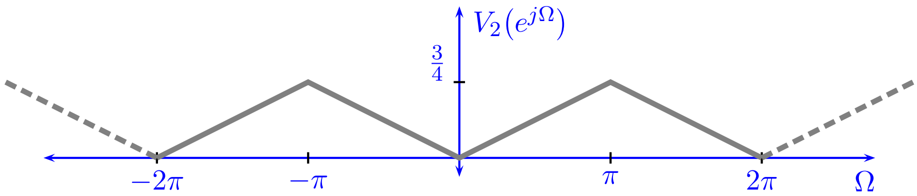


Figure 8: Frequency response of the intermediate signal $V_2(e^{j\Omega})$

(2 P)

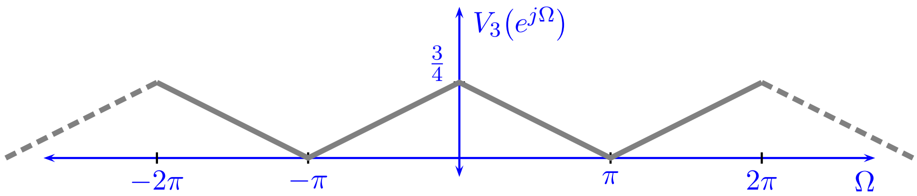


Figure 9: Frequency response of the intermediate signal $V_3(e^{j\Omega})$

(2 P)

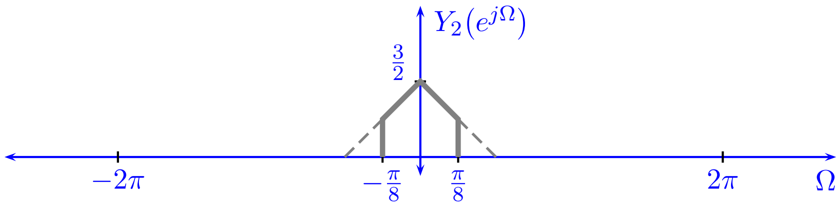


Figure 10: Frequency response of the intermediate signal $Y_2(e^{j\Omega})$

(2 P)

(iv) Draw the block diagram of a system that recovers $X_2(e^{j\Omega})$ from $V_2(e^{j\Omega})$. If you use new components sketch the corresponding frequency response of the system. Determine the maximum downsampling factor for which no aliasing occurs for $V_2(e^{j\Omega})$. (6 P)

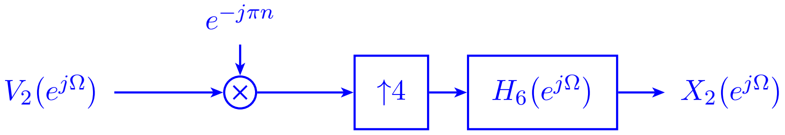
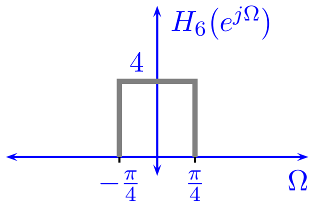


Figure 11: Block diagram of a system to recover $X_2(e^{j\Omega})$ from $V_2(e^{j\Omega})$

(4 P)



(2 P)

(b) Now, consider the multi-rate system depicted in Figure 4. The spectrum $X_3(e^{j\Omega})$ of the input signal of the system is given by Figure 5.

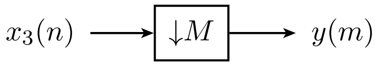


Figure 12: Block diagram of a simple multi-rate system

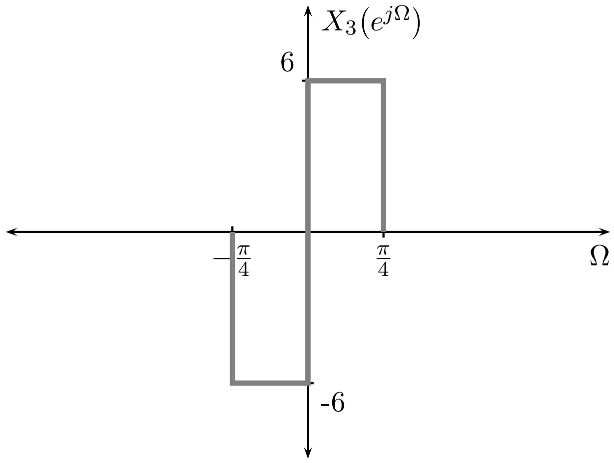
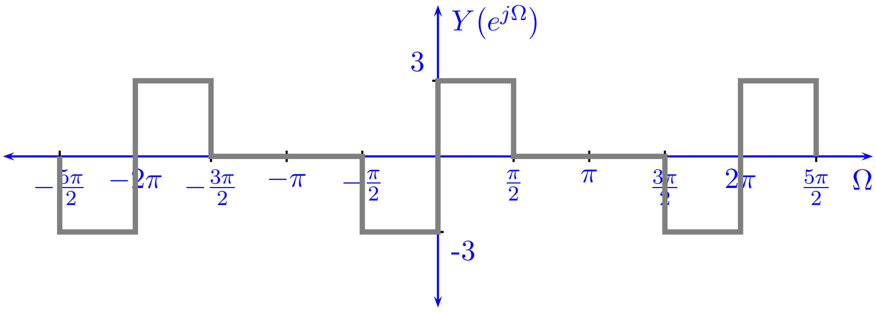


Figure 13: Input spectrum of the input signal $x_3(n)$

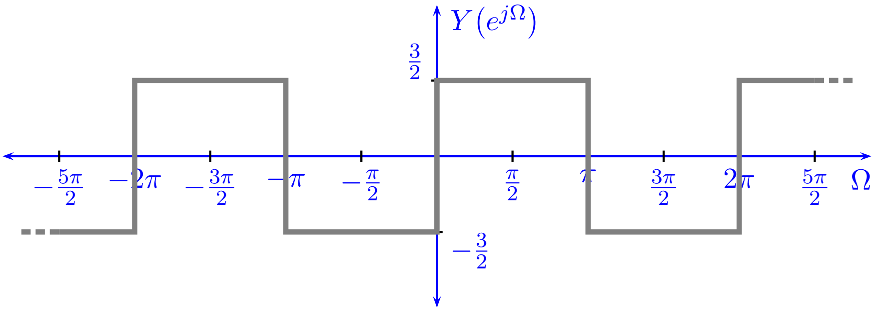
(i) Sketch $Y(e^{j\Omega})$ for $M = 2$, $M = 4$, and $M = 6$ in the range of $-2\pi < \Omega < 2\pi$. For which case(s) does aliasing occur? (7 P)

$M = 2$, no aliasing. (0.$\overline{3}$ P)



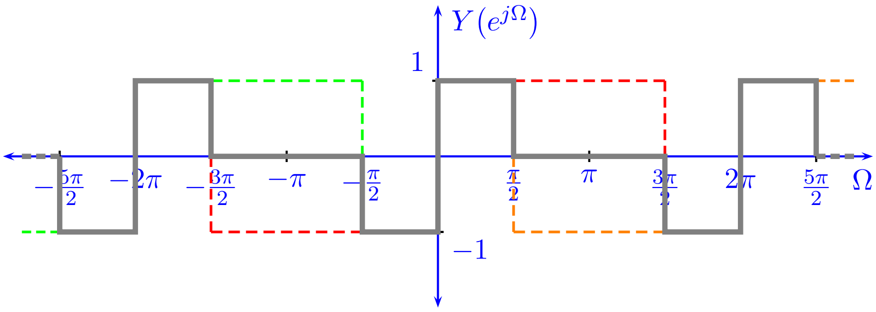
(2 P)

$M = 4$, no aliasing. (0.$\overline{3}$ P)



(2 P)

$M = 6$, aliasing. (0.$\overline{3}$ P)



(2 P)

(ii) How could aliasing be avoided in the case(s) where aliasing occurs? What drawbacks would that imply? (2 P)

To avoid aliasing a low-pass filter with cutoff frequencies $\pm\frac{\pi}{6}$ has to be used before downsampling. Of course, all information of the input signal with frequencies bigger than $\frac{\pi}{6}$ and smaller than $-\frac{\pi}{6}$ would be lost in this case. (2 P)