Solution Problem 1 (30 points)

(a) (i) When the input to this circuit is $\delta_0(t)$, the output of the summer is $\delta_0(t) - \delta_0(t-T)$. This is the input to the integrator. The impulse response of the circuit is given by

$$\begin{aligned} h(t) &= \int_0^t [\delta_0(\tau) - \delta_0(\tau - T)] d\tau \\ &= \delta_{-1}(t) - \delta_{-1}(t-T) \\ &= \operatorname{rect}\left(\frac{t - \frac{T}{2}}{T}\right) \end{aligned}$$

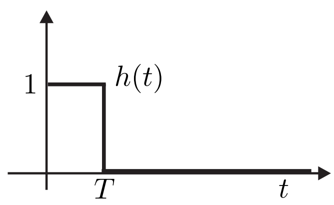


Figure 1: Impulse response of the hold circuit.

(ii) The transfer function of the circuit is

$$H(j\omega) = T \operatorname{sinc}\left(\frac{\omega T}{2} e^{-j\omega T/2}\right)$$

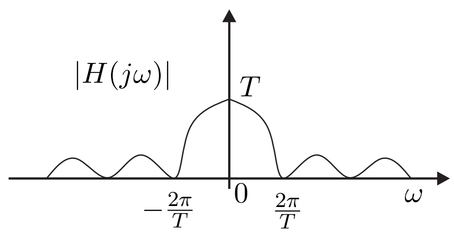


Figure 2: Magnitude frequency response of the hold circuit.

(b) Given is a signal whose length is 10 ms with a bandwdith of 10 kHz.

(i) The number of DFT points for a frequency resolution of 50 Hz is:

$$M = \frac{2 * 10000}{50} = 400$$

For a radix-2 FFT the number of FFT points has to be a power of 2 hence, the nearest number is $M = 512$ points. Hence, 512 points are necessary to compute the FFT.

(ii) The sampling rate required is

$$T = \frac{1}{f_s} = \frac{1}{20000\,\mathrm{Hz}} = 50\,\mu s$$

(iii) At $T = 50\,\mu s$ and $M = 512$, the length of the signal should be

$$L_{ms} = T \times M = 512 \times 50\,\mu s = 25.6\,\mathrm{ms}.$$

The given signal is 10 ms hence a zero padding of

$$25.6 - 10 = 15.6\,\mathrm{ms}$$

must be performed.

(c) Given

$$H(e^{j\Omega}) = \frac{e^{j\Omega} + 0.32}{e^{j2\Omega} + e^{j\Omega} + 0.16}$$

and

$$x(n) = (-0.5)^n \delta_{-1}(n)$$

We know that

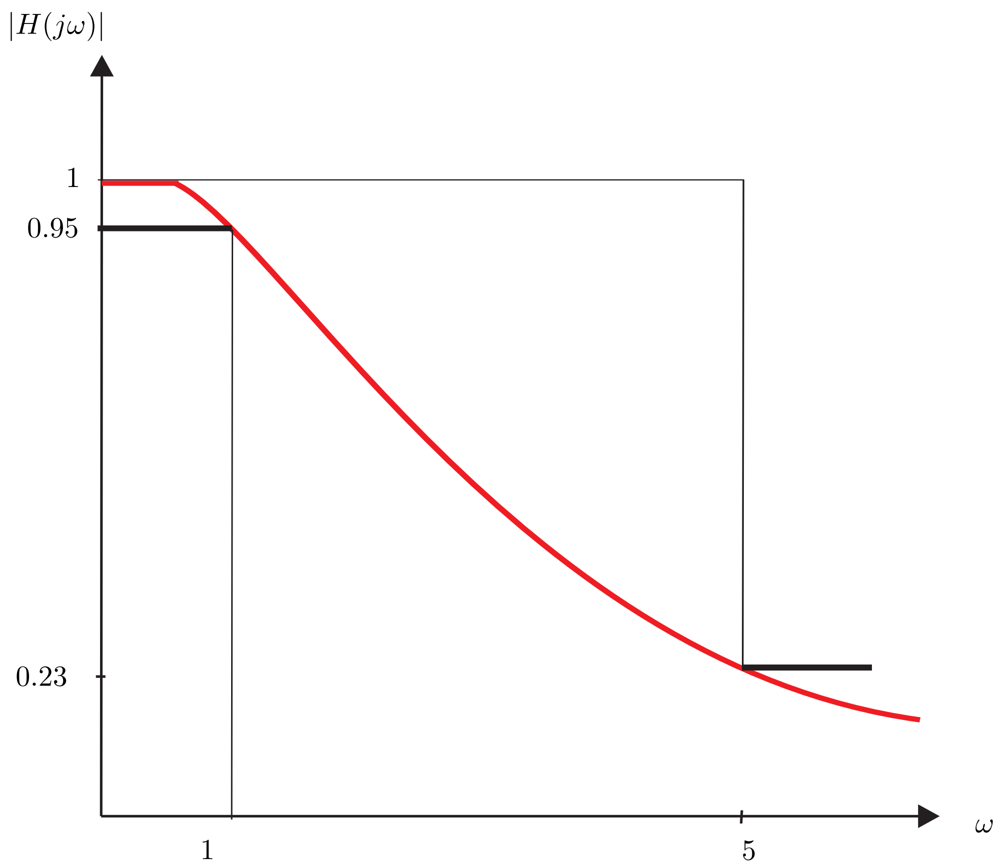
$$Y(e^{j\Omega}) = H(e^{j\Omega}) \cdot X(e^{j\Omega})$$

$$\begin{aligned}
Y(e^{j\Omega}) &= \frac{e^{j\Omega}}{e^{j\Omega} + 0.5} \cdot \frac{e^{j\Omega} + 0.32}{e^{j2\Omega} + e^{j\Omega} + 0.16} \\
\frac{Y(e^{j\Omega})}{e^{j\Omega}} &= \frac{e^{j\Omega} + 0.32}{(e^{j\Omega} + 0.5)\,(e^{j\Omega} + 0.8)\,(e^{j\Omega} + 0.2)} \\
&= \frac{2}{(e^{j\Omega} + 0.5)} - \frac{8/3}{(e^{j\Omega} + 0.8)} + \frac{2/3}{(e^{j\Omega} + 0.2)} \quad \text{(Partial fractions)} \\
Y(e^{j\Omega}) &= \frac{2e^{j\Omega}}{(e^{j\Omega} + 0.5)} - \frac{8/3e^{j\Omega}}{(e^{j\Omega} + 0.8)} + \frac{2/3e^{j\Omega}}{(e^{j\Omega} + 0.2)} \\
y(n) &= [2(-0.5)^n - 8/3(-0.8)^n + 2/3(-0.2)^n]\delta_{-1}(n)
\end{aligned}$$

Solution Problem 2 (40 points)

(a) (i) Determination of the tolerance bounds:

$$\begin{aligned}
\omega_{\mathrm{P}} = 1 &\Rightarrow H(j\omega_{\mathrm{P}}) = \frac{6.25}{(j)^2 + 4j + 6.25} = \frac{6.25}{5.25 + 4j} \\
&\Rightarrow |H(j\omega_{\mathrm{P}})| \approx 0.95 \\
&\Rightarrow \delta_1 = 1 - 0.95 = 0.05 \\
\omega_{\mathrm{S}} = 5 &\Rightarrow H(j\omega_{\mathrm{P}}) = \frac{6.25}{(5j)^2 + 20j + 6.25} = \frac{6.25}{-18.75 + 20j} \\
&\Rightarrow |H(j\omega_{\mathrm{P}})| \approx 0.23 \\
&\Rightarrow \delta_2 = 0.23
\end{aligned}$$



(ii) Determination of poles:

$$s_{1/2} = -4/2 \pm \sqrt{(4/2)^2 - 6.25} = -2 \pm 1.5j$$

Partial fraction expansion:

$$\begin{aligned}
H(s) &= \frac{A_1}{s+2+1.5j} + \frac{A_2}{s+2-1.5j} \\
A_1 &= H(s)(s+2+1.5j)|_{s=-2-1.5j} = \left.\frac{6.25}{s+2-1.5j}\right|_{s=-2-1.5j} \approx 2.08j \\
A_2 &= H(s)(s+2-1.5j)|_{s=-2+1.5j} = \left.\frac{6.25}{s+2+1.5j}\right|_{s=-2+1.5j} \approx -2.08j \\
\Rightarrow H(s) &= \frac{2.08j}{s+2+1.5j} - \frac{2.08j}{s+2-1.5j}
\end{aligned}$$

Determination of discrete poles:

$$z_{1/2} = e^{Ts_{1/2}} = e^{0.1s_{1/2}} = e^{-0.2}e^{\pm 0.15j} \approx 0.82e^{\pm 0.15j} \approx 0.81 \pm 0.12j$$

Determination of discrete system:

$$H(z) = \frac{2.08j}{1-(0.81+0.12j)z^{-1}} - \frac{2.08j}{1-(0.81-0.12j)z^{-1}}$$

Determination of impulse response by using a mathematical tabular:

$$\begin{aligned}
h(n) &= 2.08je^{-0.2n}e^{-j0.15n}\gamma_{-1}(n) - 2.08je^{-0.2n}e^{j0.15n}\gamma_{-1}(n) \\
&= -2.08je^{-0.2n}\left(e^{j0.15n} - e^{-j0.15n}\right)\gamma_{-1}(n) \\
&= -2.08je^{-0.2n}2j\sin(0.15n)\gamma_{-1}(n) \\
&= 4.16e^{-0.2n}\sin(0.15n)\gamma_{-1}(n)
\end{aligned}$$

(b) (i) 1. Constant signals are suppressed totally:

$$\Rightarrow G\left(z = e^{j0}\right) = 0$$

2. Signals with half of the sampling frequency are suppressed totally as well:

$$\Rightarrow G\left(z = e^{j\pi}\right) = 0$$

3. The system has the minimum number of zeros, that are necessary to fulfill the characteristics 1 and 2:

The system has two zeros:

$$\Rightarrow z_{0,1} = 1 \quad \text{and} \quad z_{0,2} = -1$$

The system is causal and non recursive:

$$\Rightarrow G(z) = K\frac{(z-1)(z+1)}{z^2} = K\frac{(z^2-1)}{z^2} = K\left(1 - z^{-2}\right)$$

4. The system is a bandpass with a maximum system gain of 4.

$$\begin{aligned}\Rightarrow |G(j\omega_{\text{max}})| &= \left|G\left(j\frac{\pi}{2}\right)\right| \\ &= \left|K\left(1-(-j)^2\right)\right| \overset{!}{=} 4 \\ \Rightarrow K &= \pm 2 \\ \Rightarrow G(z) &= \pm 2\left(1 - z^{-2}\right)\end{aligned}$$

(ii) Difference equation can be read directly from the transfer function:

$$y(n) = \pm 2(v(n) - v(n-2))$$

(c) (i) Rewriting of the impulse response leads to:

$$h_{\text{H}}(n) = \frac{1-\cos(\pi n)}{\pi n} = \begin{cases} \frac{2}{n\pi}, & \text{for } n \text{ being odd,} \\ 0, & \text{else} \end{cases}.$$

Determination of the transfer function:

$$\begin{aligned}H_{\text{S,H}}(e^{j\Omega}) &= \sum_{n=-5}^{n=5} h_{\text{S,H}}(n)e^{j\Omega n} \\ &= \frac{2}{\pi}\left(e^{-j\Omega} - e^{j\Omega}\right) + \frac{2}{3\pi}\left(e^{-j3\Omega} - e^{j3\Omega}\right) + \frac{2}{5\pi}\left(e^{-j5\Omega} - e^{j5\Omega}\right) \\ &= -j\frac{4}{\pi}\left(\sin(\Omega) + \frac{1}{3}\sin(3\Omega) + \frac{1}{5}\sin(5\Omega)\right)\end{aligned}$$

(ii) To make the impulse response causal a delay or $\kappa = 5$ samples is necessary. Therefore the impulse response $h_{\mathrm{C}}(n)$ of the causal system is:

$$h_{\mathrm{C}}(n) = h_{\mathrm{H}}(n-5)$$

(iii) Using the shift theorem leads to:

$$\begin{aligned} h_{\mathrm{C}}(n) &= h_{\mathrm{H}}(n-5) \\ \Rightarrow H_{\mathrm{C}}(e^{j\Omega}) &= H_{\mathrm{S,H}}(e^{j\Omega})e^{-j5\Omega} \\ &= \frac{4}{\pi}\left(\sin(\Omega) + \frac{1}{3}\sin(3\Omega) + \frac{1}{5}\sin(5\Omega)\right)e^{-j\frac{\pi}{2}}e^{-j5\Omega} \\ &= A(\Omega)e^{-j(5\Omega+\frac{\pi}{2})} \text{ , with } A(\Omega) \in \mathbb{R} \end{aligned}$$

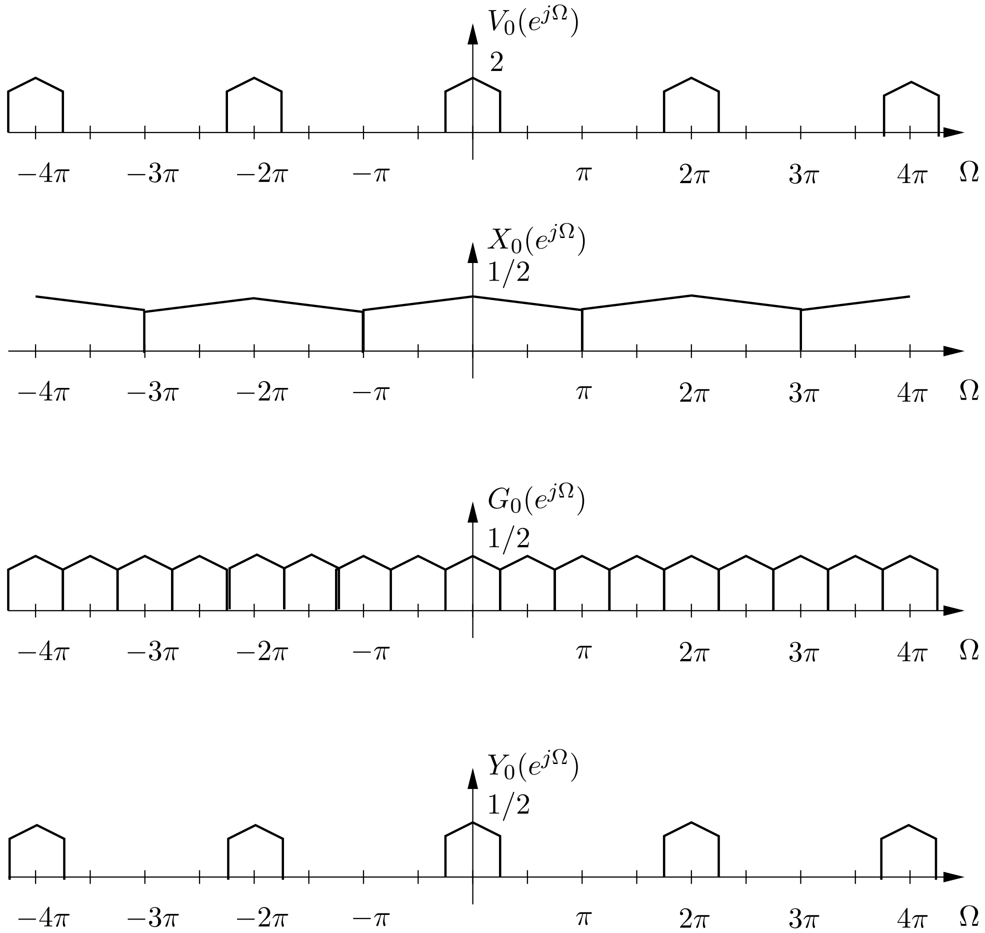
(iv)

$$\phi(\Omega) = \begin{cases} -5\Omega - \frac{\pi}{2}, & \text{for } A(\Omega) \geq 0, \\ -5\Omega + \frac{\pi}{2} & \text{else} \end{cases}.$$

(v) The impulse response $h_{\mathrm{C}}(n)$ is symmetric to $n = 5 \Rightarrow$ The shifted, approximated Hilbert transformer is a linear phase system.

Solution Problem 3 **(30 points)**

(a) (i)



(ii) $V_0(e^{j\Omega}) = H_0(e^{j\Omega}) \cdot X(e^{j\Omega})$

$$\begin{aligned} X_0(e^{j\Omega}) &= \frac{1}{M}\sum_{k=0}^{M-1} V_0(e^{j(\Omega-k2\pi)/M}) \\ &= \frac{1}{4}\Big[X(e^{j\frac{\Omega}{4}})H_0(e^{j\frac{\Omega}{4}}) + X(e^{j(\frac{\Omega}{4}-\frac{\pi}{2})})H_0(e^{j(\frac{\Omega}{4}-\frac{\pi}{2})}) \\ &\quad +X(e^{j(\frac{\Omega}{4}-\pi)})H_0(e^{j(\frac{\Omega}{4}-\pi)}) + X(e^{j(\frac{\Omega}{4}-\frac{3\pi}{2})})H_0(e^{j(\frac{\Omega}{4}-\frac{3\pi}{2})})\Big] \end{aligned}$$

$$\begin{aligned} G_0(e^{j\Omega}) &= X_0(e^{jL\Omega}) \\ &= \frac{1}{4}\Big[X(e^{j\Omega})H_0(e^{j\Omega}) + X(e^{j(\Omega-\frac{\pi}{2})})H_0(e^{j(\Omega-\frac{\pi}{2})}) \\ &\quad +X(e^{j(\Omega-\pi)})H_0(e^{j(\Omega-\pi)}) + X(e^{j(\Omega-\frac{3\pi}{2})})H_0(e^{j(\Omega-\frac{3\pi}{2})})\Big] \end{aligned}$$

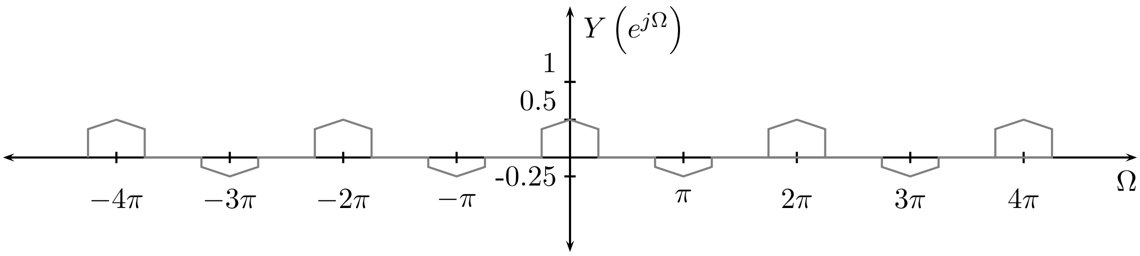
(iii)

$$Y_0(e^{j\Omega}) = G_0(e^{j\Omega}) \cdot H_0(e^{j\Omega})$$

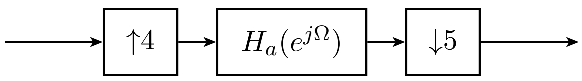
$Y_1(e^{j\Omega})$ can be derived from $Y_1(e^{j\Omega})$ because of the relation of $H_0(e^{j\Omega})$ and $H_1(e^{j\Omega})$.

$$\begin{aligned} Y(e^{j\Omega}) &= Y_0(e^{j\Omega}) - Y_1(e^{j\Omega}) \\ &= \tfrac{1}{4}X(e^{j\Omega})\left[H_0^2(e^{j\Omega}) - H_0^2(e^{j(\Omega+\pi)})\right] \\ &\quad +\tfrac{1}{4}X(e^{j(\Omega-\frac{\pi}{2})})\left[H_0(e^{j\Omega})H_0(e^{j(\Omega-\frac{\pi}{2})}) - H_0(e^{j(\Omega+\pi)})H_0(e^{j(\Omega+\frac{\pi}{2})})\right] \\ &\quad +\tfrac{1}{4}X(e^{j(\Omega-\frac{3\pi}{2})})\left[H_0(e^{j\Omega})H_0(e^{j(\Omega-\frac{3\pi}{2})}) - H_0(e^{j(\Omega+\pi)})H_0(e^{j(\Omega-\frac{\pi}{2})})\right] \end{aligned}$$

(iv)



(b) A sample rate conversion from 10 kHz to 8 kHz requires a non-integer downsampling by a factor of $\frac{10}{8}$. This can be achieved by first upsampling with a factor of 4 and then downsampling with a factor of 5. In order to avoid aliasing effects, an anti-aliasing filter $H_a(e^{j\Omega})$ has to be used after the upsampling process.



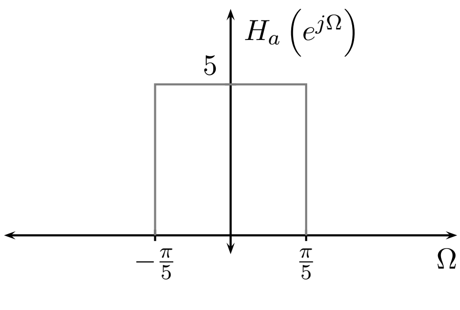


Figure 3: Fourier spectra

(c) To avoid aliasing the filter $H_0\left(e^{j\Omega}\right)$ has to be a lowpass-filter with cutoff-frequency $\Omega_c = \frac{\pi}{L}$.