Advanced Digital Signal Processing

Examination SS 2017

Examiner: Prof. Dr.-Ing. Gerhard Schmidt

Date: 14.09.2017

Name: ______________________________

Matriculation Number: ______________________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorised to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ______________________________

Marking

Problem	1	2	3
Points	/30	/25	/45

Total number of points: __________________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated __________________ Signature: ______________________________

Advanced Digital Signal Processing

Examination SS 2017

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	14.09.2017
Time:	09:00 h – 10:30 h (90 minutes)
Location:	KS2, C-SR I

Remarks

- Please check that you have received a cover sheet plus 4 sheets with 3 problems.
- Please write your **name** and your **matriculation number** on each sheet of paper that you return.
- Please keep your student ID and your identity card ready.
- During the exam only questions concerning the problems are answered.
- Please don't use any pencil or red pen.
- Please use a **new** sheet of paper with your name and matriculation number on it for **each problem**. You can ask for more sheets of paper, if necessary.
- The exam is open books, open notes; other people are closed. Programmable electronical devices except pocket calculators are not permitted.
- Partial credit will be given. No credit will be given if an answer appears with no supporting work or reason.
- Note that the given points of the subproblems are just preliminary.
- At the end of the exam put all sheets together as you have received them, including the problem sheets.
- No one is allowed to talk or to leave his or her seat until **all** exams have been collected.
- The problems and the solutions will be published on the website of the lecture. Also the date and the place of the inspection will be announced on this website.

Problem 1 (30 points)

This problem consists of four parts (a), (b), (c) and (d). They are **not** related to each other and can be solved **independently**.

(a) The period of a periodic discrete time signal is 0.25 milliseconds. Each period is sampled at 50 equally spaced points. It is assumed that with this number of samples, the sampling theorem is satisfied and thus there will be no aliasing.

(i) Compute the sampling period T. (1 P)

$$\begin{aligned} T_s &= \frac{\text{Signal period}}{\text{Number of samples}} \\ &= \frac{0.25 \cdot 10^{-3}}{50}\,\mathrm{s} \\ &= 5 \cdot 10^{-6}\,\mathrm{s} \\ &= 5\,\mu\mathrm{s}. \end{aligned}$$

(ii) Compute the sampling frequency f_s. (1 P)

$$\begin{aligned} f_s &= \frac{1}{T_s} \\ &= 2 \cdot 10^{5}\ \mathrm{Hz} \\ &= 200\,\mathrm{kHz} \end{aligned}$$

(b) A sinusoid signal $x(t)$ with a RMS-Amplitude of $10\mathrm{V}/\sqrt{2}$ can be quantized with a 24-bit A/D converter containing a linear quantizer without clipping.

(i) Determine: The possible range R of the input signal for optimal A/D level control, the quantization levels L the quantizer has, and the optimal quantization step Δ. (2 P)
Range of the signal: $R = \pm 10\mathrm{V} = 20\,\mathrm{V}$
Number of quantization levels: $L = 2^b = 2^{24} = 16777216$
Quantization step: $\Delta = R/L = 20\mathrm{V}/16777216 \approx 1.19\,\mu\mathrm{V}$

(ii) Sketch the mathematical model of the system with the added quantization noise after the sampling process $x(n) = x(n \cdot T)$. (1 P)
The mathematical model is shown below:

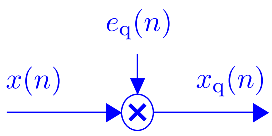


Figure 1: Mathematical model of the quantizer

(iii) Compute the variance of the quantization error when the probability distribution of the quantization error can be assumed to be uniformly distributed and has zero mean. (2 P)

The variance of the quantization error for a uniform pdf input is given by $\sigma_e^2 = \frac{\Delta^2}{12} = \frac{(1.19\mu\mathrm{V})^2}{12} \approx 1.184 \cdot 10^{-13}\,\mathrm{V}^2$

(c) Given is a sampled signal $x(n)$ that is time limited to 10 ms and has an essential bandwidth of 20 kHz.

(i) For the case where the number of signal points in the sampled version of the signal is equal to the number of FFT points ($L = M$) determine L the number of signal samples necessary to compute a radix-2 FFT with a frequency resolution (bin distance) of at least 50 Hz. (4 P)

$$L = \frac{2 * 10000}{25} = 800$$

For a radix-2 FFT the number of FFT points has to be a power of 2 hence, the nearest number is $M = 1024$ points. Hence, 1024 points are necessary to compute the FFT.

(ii) What is the maximum sampling interval at which the signal must be sampled in order to retain the information present within the 20 kHz bandwidth. (2 P)

The following sampling rate is required:

$$T = \frac{1}{f_s} = \frac{1}{40000\,\mathrm{Hz}} = 25\,\mu s$$

(iii) Does the signal need to be zero padded? If yes, for how much duration (in ms)? Explain your answer. At $T = 25\,\mu s$ and $M = 1024$, the length of the signal should be (4 P)

$$L_{ms} = T \times M = 1024 \times 25\,\mu s = 25.6\,\mathrm{ms}.$$

The given signal is 10 ms hence a zero padding of

$$25.6 - 10 = 15.6\,\mathrm{ms}$$

must be performed.

(iii) Find the 4-point DFT $X(\mu)$ for an input signal $x(n) = \{1, 1, -1, 5\}$ and $\mu = 0 \ldots 3$. (3 P)

$X(0) = 1 + 1 - 1 + 5 = 6$
$X(1) = 1 - j + 1 + 5j = 2 + 4j$
$X(2) = 1 - 1 - 1 - 5 = -6$
$X(3) = 1 + j + 1 - 5j = 2 - 4j$
$X(\mu) = \{6, 2 + 4j, -6, 2 - 4j\}$

(d) Find the response $y(n)$ of a causal system with the frequency response (10 P)

$$H(e^{j\Omega}) = \frac{e^{j\Omega} + 0.32}{e^{j2\Omega} + e^{j\Omega} + 0.16}$$

by using the Fourier transform to the input

$$x(n) = (-0.5)^n \delta_{-1}(n),$$

with

$$\delta_{-1}(n) = \begin{cases} 1, \text{ for } \ n \geq 0, \\ 0, \text{ else.} \end{cases}$$

Given $H(e^{j\Omega}) = \frac{e^{j\Omega}+0.32}{e^{j2\Omega}+e^{j\Omega}+0.16}$ and $x(n) = (-0.5)^n \delta_{-1}(n)$.

We know that

$$Y(e^{j\Omega}) = H(e^{j\Omega}) \cdot X(e^{j\Omega})$$

$Y(e^{j\Omega}) = \frac{e^{j\Omega}}{e^{j\Omega}+0.5} \cdot \frac{e^{j\Omega}+0.32}{e^{j2\Omega}+e^{j\Omega}+0.16}$

$\frac{Y(e^{j\Omega})}{e^{j\Omega}} = \frac{e^{j\Omega}+0.32}{(e^{j\Omega}+0.5)\,(e^{j\Omega}+0.8)\,(e^{j\Omega}+0.2)}$

$\frac{Y(e^{j\Omega})}{e^{j\Omega}} = \frac{2}{(e^{j\Omega}+0.5)} - \frac{8/3}{(e^{j\Omega}+0.8)} + \frac{2/3}{(e^{j\Omega}+0.2)}$ (Partial fractions)

$Y(e^{j\Omega}) = \frac{2e^{j\Omega}}{(e^{j\Omega}+0.5)} - \frac{8/3e^{j\Omega}}{(e^{j\Omega}+0.8)} + \frac{2/3e^{j\Omega}}{(e^{j\Omega}+0.2)}$

$y(n) = [2(-0.5)^n - 8/3(-0.8)^n + 2/3(-0.2)^n]\delta_{-1}(n)$

Problem 2 (25 points)

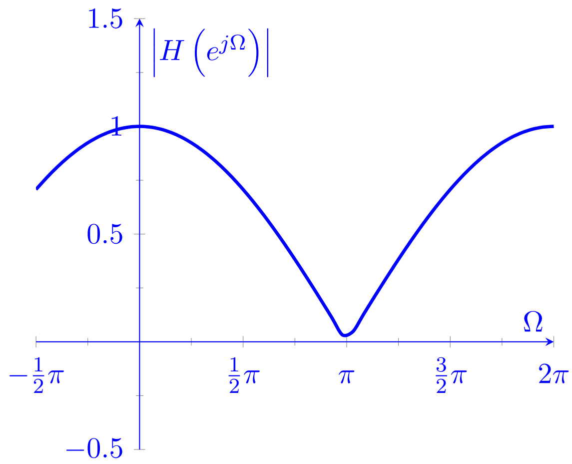
Given is a discrete signal $v(n) = \cos(\Omega_0 n)$. The corresponding continuous signal $v(t)$ was sampled with $f_s = \frac{1}{T_s} = 40$ kHz. $v(n)$ shall be decimated by a factor of $M = 2$. A anti aliasing filter $H(z)$ with

$$H(z) = \frac{1 + z^{-1}}{2}$$

is used before the signal gets down sampled.

(a) Determine and sketch the amplitude response $\left|H\left(e^{j\Omega}\right)\right|$ of the anti aliasing filter for $\Omega \in [0...2\pi]$. Label all axes. (8 P)

$$H\left(e^{j\Omega}\right) = \frac{(1 + e^{-j\Omega})}{2} = \frac{e^{\frac{-j\Omega}{2}}(e^{\frac{j\Omega}{2}} + e^{\frac{-j\Omega}{2}})}{2} = \cos(\Omega/2)e^{\frac{-j\Omega}{2}}$$



(b) Sketch the whole system with $V(\mathrm{e}^{j\Omega})$ being the input, $X(\mathrm{e}^{j\Omega})$ being the output of the anti aliasing filter and $Y(\mathrm{e}^{j\Omega})$ being the output of the down sampler. (3 P)

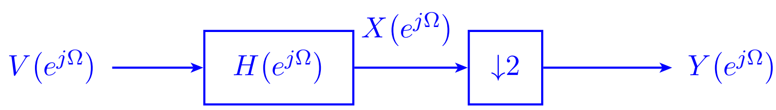


Figure 2: Block diagram of a multi-rate system.

(c) Determine the amplitude spectrum of $X(e^{j\Omega})$ for $\Omega_0 = \frac{3\pi}{4}$. (2 P)

$$
\begin{aligned}
|X(e^{j\Omega})| &= |H(e^{j\Omega})| \cdot |\mathcal{F}\{v(n)\}| \\
&= \cos\left(\frac{3\pi}{8}\right)\pi \sum_{\lambda=-\infty}^{\infty}\left[\delta_0\left(\Omega - 2\pi\lambda + \frac{3\pi}{4}\right) + \delta_0\left(\Omega - 2\pi\lambda - \frac{3\pi}{4}\right)\right] \\
&= 0.38\pi \sum_{\lambda=-\infty}^{\infty}\left[\delta_0\left(\Omega - 2\pi\lambda + \frac{3\pi}{4}\right) + \delta_0\left(\Omega - 2\pi\lambda - \frac{3\pi}{4}\right)\right]
\end{aligned}
$$

Now another signal

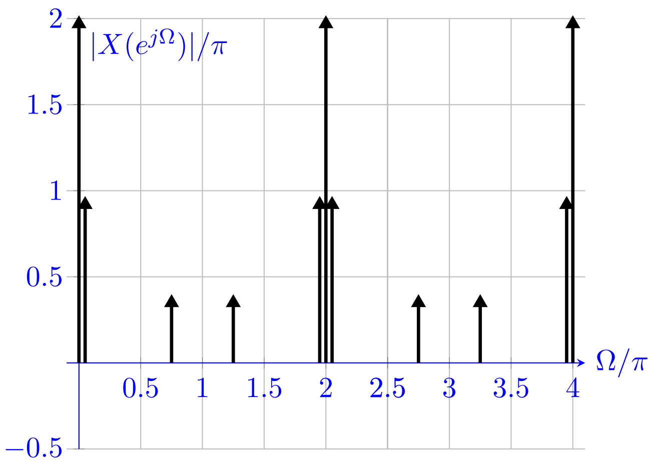
$$u(n) = 1 + \cos(\Omega_0 n) + \cos(\Omega_1 n)$$

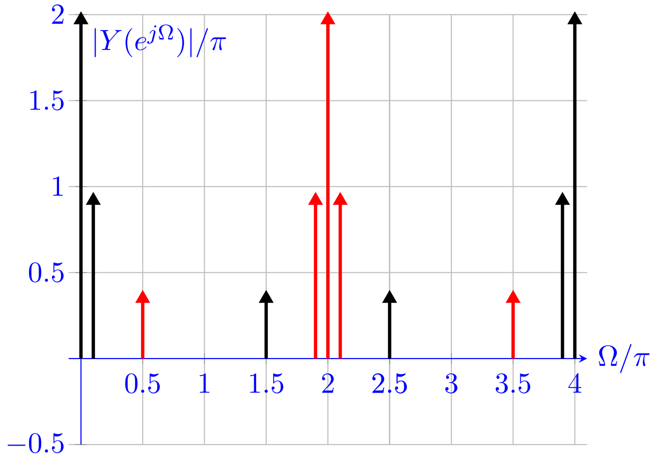
with $\Omega_0 = \frac{3\pi}{4}$ and $\Omega_1 = \frac{\pi}{20}$is given to the system. The sample frequency used to sample the corresponding signal $u(t)$ is $f_s = \frac{1}{T_s} = 40$ kHz.

(d) What is the new sample frequency $f_{s,M}$ of $Y(e^{j\Omega})$? (2 P)

$$f_{s,M} = \frac{f_s}{M} = 20 \text{ kHz}$$

(e) Sketch the amplitude spectra $|X(e^{j\Omega})|$ and $|Y(e^{j\Omega})|$ for $\Omega \in [0...4\pi]$. Label all axis. (7 P)





(e) Explain which frequency components are existing in $|Y(e^{j\Omega})|$ and why that happend. (3 P)

There is one component at $\Omega = \frac{\pi}{10}$ which is wanted and an unwanted component at $\Omega = \frac{\pi}{2}$. This one results of the poor stopband attenuation of the first order anti aliasing filter.

Problem 3 (45 points)

This problem consists of two parts (a) and (b). They are **not** related to each other and can be solved **independently**.

(a) Consider the signal flow chart given in Figure 1 where $v(n)$ is the input of the system and $y(n)$ the corresponding output.

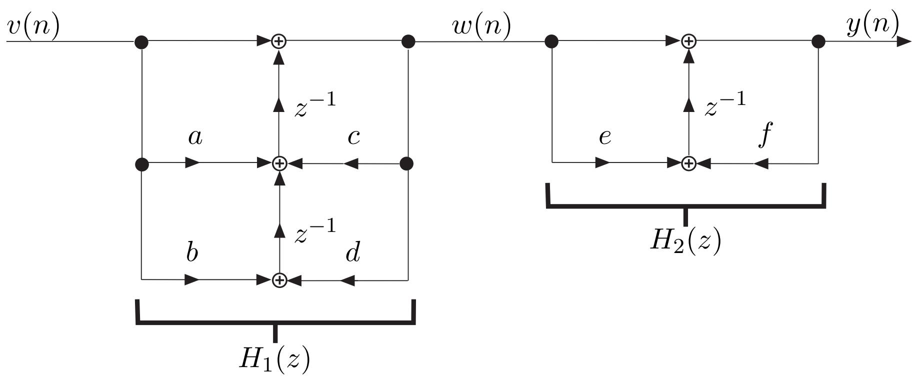


Figure 3: Signal flow chart.

(i) Of which structure is in the overall system? The overall system has a serial/-cascade structure. (1 P)

(ii) In which form are the systems $H_1(z)$ and $H_2(z)$ given? The systems $H_1(z)$ and $H_2(z)$ are given in direct form II. (1 P)

Suppose from now on the following coefficients for the system depicted in Figure 1.

$$a = \frac{4}{3} \qquad b = -\frac{4}{3} \qquad c = -\frac{1}{4} \qquad d = \frac{3}{8} \qquad e = -\frac{1}{2}$$

(iii) Choose the parameter f such that the system H_2 gets an allpass. For an allpass $z_{0,i} = -z_{\infty,i}*$ should hold, so $e = -f$, which leads to $f = \frac{1}{2}$. (2 P)

(iv) Determine the transfer function $H_{tot}(z)$ of the overall system. From the flow chart one can read: (6 P)

$$H_1(z) = \frac{1 + \frac{4}{3}z^{-1} - \frac{4}{3}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$$

and

$$H_2(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

Since the overall system is given in cascaded form one can get the transfer function by:

$$\begin{aligned} H_{tot}(z) &= H_1(z) \cdot H_2(z) \\ &= \frac{1+\frac{4}{3}z^{-1}-\frac{4}{3}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} \cdot \frac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}} \\ &= \frac{1+\frac{5}{6}z^{-1}-2z^{-2}+\frac{2}{3}z^{-3}}{1+\frac{3}{4}z^{-1}-\frac{1}{4}z^{-2}-\frac{3}{16}z^{-3}} \\ &= \frac{z^3+\frac{5}{6}z^2-2z+\frac{2}{3}}{z^3+\frac{3}{4}z^2-\frac{1}{4}z-\frac{3}{16}} \end{aligned}$$

(v) Derive the difference equation for $y(n)$ depending on $v(n)$. (3 P)
The transfer function is defined as:

$$H_{tot}(z) = \frac{Y(z)}{V(z)}$$

So the difference equation can be determined by rewriting the solution from the previous problem and transforming the result back into the time domain:

$$Y(z)+\frac{3}{4}Y(z)z^{-1}-\frac{1}{4}Y(z)z^{-2}-\frac{3}{16}Y(z)z^{-3} = V(z)+\frac{5}{6}V(z)z^{-1}-2V(z)z^{-2}+\frac{2}{3}V(z)z^{-3}$$

$$\bullet\!\!-\!\!\circ$$

$$y(n)+\frac{3}{4}y(n-1)-\frac{1}{4}y(n-2)-\frac{3}{16}y(n-3) = v(n)+\frac{5}{6}v(n-1)-2v(n-2)+\frac{2}{3}v(n-3)$$

(vi) Determine the poles and zeros of the overall system and sketch the pole-zero diagram. Poles: (8 P)
The first pole is given by system H_2: $z_{\infty,1} = -\frac{1}{2}$.
The other poles can be determined by:

$$z^2+\frac{1}{4}z-\frac{3}{8} = 0$$

$$\begin{aligned} z_{\infty,2/3} &= -\frac{1}{8} \pm \sqrt{\frac{1}{64}+\frac{3}{8}} \\ &= -\frac{1}{8} \pm \sqrt{\frac{1}{64}+\frac{24}{64}} \\ &= -\frac{1}{8} \pm \sqrt{\frac{25}{64}} \\ &= -\frac{1}{8} \pm \frac{5}{8} \end{aligned}$$

$$\begin{aligned} z_{\infty,2} &= \frac{1}{2} \\ z_{\infty,3} &= -\frac{3}{4} \end{aligned}$$

Zeros:
The first zero is given by system H_2: $z_{0,1} = \frac{1}{2}$.
The other zeros can be determined by:

$$z^2 + \frac{4}{3}z - \frac{4}{3} = 0$$

$$\begin{aligned} z_{0,2/3} &= -\frac{2}{3} \pm \sqrt{\frac{4}{9} + \frac{4}{3}} \\ &= -\frac{2}{3} \pm \sqrt{\frac{4}{9} + \frac{12}{9}} \\ &= -\frac{2}{3} \pm \sqrt{\frac{16}{9}} \\ &= -\frac{2}{3} \pm \frac{4}{3} \end{aligned}$$

$$\begin{aligned} z_{0,2} &= \frac{2}{3} \\ z_{0,3} &= -2 \end{aligned}$$

Thus, the overall system has the following pole-zero diagram:

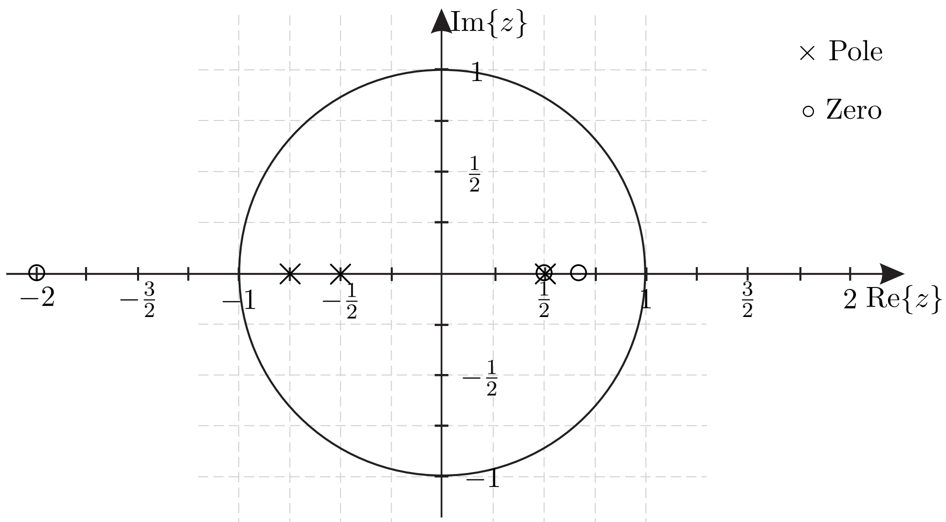


Figure 4: Pole-zero diagram.

(vii) Is the overall system stable? Give reason to your answer. The overall system is stable, since all poles $z_{\infty,1-3}$ lie inside the unit circle. (3 P)

(viii) What kind of filter is represented by the overall system (bandpass, lowpass, highpass, etc.)? Give reason to your answer. The overall system is a highpass filter. This can be seen in the pole-zero diagram from problem (vi): One zero (3 P) at $\frac{1}{2}$ is compensated by a pole at the same condition. On the other hand the zero at -2 is compensated by a pole at $-\frac{1}{2}$, which fulfills an allpass condition. Thus the remaining zero at $\frac{2}{3}$ causes a damping of the lower frequencies and the remaining pole at $\frac{3}{4}$ causes a boost of the higher frequencies, consequently the system is an highpass filter.

(b) A continuous lowpass filter with the following transfer function is given:

$$H_c(s) = \frac{1}{s^2 + 4s + 3}.$$

(i) This filter should be used as the basis for designing a digital filter $H_d(z)$. Which methods can be used to convert an analog filter design into a digital filter? In the lecture the impulse invariant transform and the bilinear transform were presented to convert an analog filter into a digital filter. (2 P)

(ii) Determine the transfer function $H_d(z)$ using the bilinear transform with $T_s = 1$. $H_c(s)$ is given by: (6 P)

$$H_c(s) = \frac{1}{s^2 + 4s + 3}.$$

Substitution of s by

$$s = \frac{2}{T_s}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right),$$

with $T_s = 1$ we get

$$s = 2\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right).$$

We get for the $\mathcal{Z}$-Domain transfer function $H_d(z)$:

$$\begin{aligned} H_d(z) &= \frac{1}{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 8\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 2} \\ &= \frac{(1+z^{-1})^2}{4(1-z^{-1})^2 + 8(1-z^{-1})(1+z^{-1}) + 2(1+z^{-1})^2} \\ &= \frac{1 + 2z^{-1} + z^{-2}}{4 - 8z^{-1} + 4z^{-2} - 8 + 8z^{-2} + 2 + 4z^{-1} + 2z^{-2}} \\ &= \frac{1 + 2z^{-1} + z^{-2}}{-2 - 4z^{-1} + 14z^{-2}} \\ &= \frac{z^2 + 2z + 1}{-2z^2 - 4z + 14} \end{aligned}$$

(iii) Which properties does the bilinear transform have? (3 P)

- The stability of the system is preserved.
- The $j\omega$-axis in the s-plane is mapped to one revolution of the unit circle in the z-plane.
- The transform is unique.
- The kind of filter is preserved (bandpass, lowpass, highpass, etc.).

(iv) Why is the above described procedure of designing an analog filter and subsequent transformation to a digital filter useful? Starting with designing an analog filter instead of a digital filter directly is useful, since analog filter design is a well developed field. It gives a lot of existing design catalogs, already. For implementation purposes then the transformation to a digital filter is necessary. (2 P)

(v) Name two commonly used filter types that were treated in the lecture for designing lowpass filters. Which are the differences in the frequency response of these types? A first example is the butterworth filter. Its frequency response has the property, that it meets the specification very well at the beginning and at the end but it is worse than other design methods in the transition from passband to stopband. (5 P)

A second example is the Chebychev filter. Depending on which type of Chebychev filter is used it has a worse behavior in the pass- or stopband, which is expressed in a higher ripple of the frequency response. But the behavior in the transition from passband to stopband is better, due to a higher slope of the frequency response.