Advanced Digital Signal Processing

Examination SS 2016

Examiner: Prof. Dr.-Ing. Gerhard Schmidt

Date: 15.09.2016

Name: ______________________________

Matriculation Number: ______________________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorised to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ______________________________

Marking

Problem	1	2	3
Points	/30	/30	/40

Total number of points: ________________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated ________________ Signature: ______________________________

Advanced Digital Signal Processing

Examination SS 2016

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	15.09.2016
Time:	09:00 h – 10:30 h (90 minutes)
Location:	KS2, C-SR I

Remarks

- Please check that you have received a cover sheet plus 4 sheets with 3 problems.
- Please write your **name** and your **matriculation number** on each sheet of paper that you return.
- Please keep your student ID and your identity card ready.
- During the exam only questions concerning the problems are answered.
- Please don't use any pencil or red pen.
- Please use a **new** sheet of paper with your name and matriculation number on it for **each problem**. You can ask for more sheets of paper, if necessary.
- The exam is open books, open notes; other people are closed. Programmable electronical devices except pocket calculators are not permitted.
- Partial credit will be given. No credit will be given if an answer appears with no supporting work or reason.
- Note that the given points of the subproblems are just preliminary.
- At the end of the exam put all sheets together as you have received them, including the problem sheets.
- No one is allowed to talk or to leave his or her seat until **all** exams have been collected.
- The problems and the solutions will be published on the website of the lecture. Also the date and the place of the inspection will be announced on this website.

Problem 1 (30 points)

This question consists of three parts (a), (b) and (c). They are **not** related to each other and can be solved independently.

(a) Given is the following continuous-time signal $x_a(t)$:

$$x_a(t) = \ 3 \cdot \sin(2 \cdot \pi \cdot 50\,\text{Hz} \cdot t)$$

(i) Sketch the continuous-time signal $x_a(t)$ in the time interval from 0 to 30 ms. (2 P)

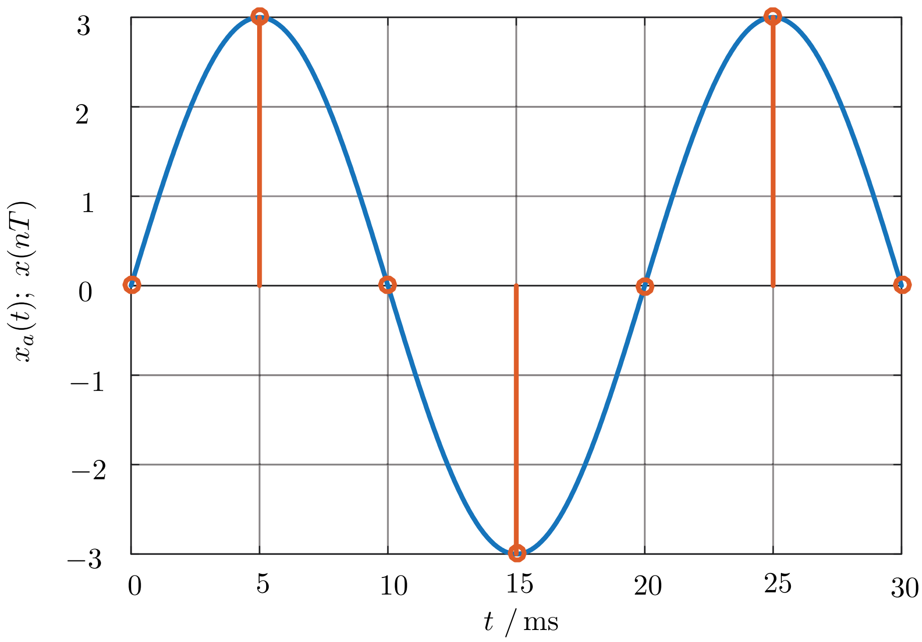


Figure 1: Solutions for (a,i) and (a,iv)

(ii) Can you find a sampling rate f_s such that the discrete-time signal $x(n) = x_a(nT)$, $T = 1/f_s$, reaches its peak value of 3? What is the minimum f_s suitable for this task? (3 P)

$x\left(1\right) \overset{!}{=} 3$

$3 = 3 \cdot \sin\left(2 \cdot \pi \cdot \frac{50\,\text{Hz}}{f_s}\right) \Rightarrow f_s = 200\,\text{Hz}$

(iii) Determine the frequency of the discrete-time signal and show that the signal is periodic. (3 P)

$$x\left(n\right) = x_a\left(nT\right) = x_a\left(n/f_s\right) = 3 \cdot \sin\left(2 \cdot \pi \cdot \frac{50Hz}{200Hz} \cdot n\right) = 3 \cdot \sin\left(\frac{\pi}{2} \cdot n\right)$$

$$f = \frac{1}{2\pi}\left(\frac{\pi}{2}\right) = \frac{1}{4}, N_p = 4$$

(iv) Compute the sampled values in one period of $x(n)$. Sketch the discrete-time signal $x(n)$ in the same diagram with $x_a(t)$. (4 P)

$y(n) = \{0, 3, 0, -3\}$ Refer to Figure 1

(b) In the following consider a sinusoid signal $v(n) = 5\,\text{V} \cdot \sin\left(\frac{\omega_0}{\omega_s} \cdot n\right)$ with $f_0 = 5$ Hz and $f_s = 10$ kHz which has to be quantized with a midrise quantizer. The word length of the quantizer is 4 bits. The quantizer has a digital full scale.

(i) Determine the range R of the signal, the quantization levels L the quantizer has, and the quantization step Δ. (3 P)

Range of the signal $R = 10\,\text{V}([-5V, 5V])$, Number of quantization levels L (2 P)

$$L = 2^b = 2^4 = 16$$

Quantization step Δ (1 P)

$$\Delta = R/L = 10/16 = 0.625\,\text{V}$$

(ii) Calculate the power P_n of the quantization noise. (1 P)

(1 P)

$$\begin{aligned} P_\text{n} &= {\sigma_{eq}}^2 = \Delta^2/12 \\ &= \frac{(0.625)^2}{12} \\ &= 0.03255 \end{aligned}$$

(iii) Determine the SNR in dB and in linear scale. (3 P)

$\text{SNR/dB} = 6.02b + 10.8 - 20\log_{10}(R/\sigma_\text{v})$
R is the range of the quantizer
σ_v is the RMS amplitude of the input signal
$R = 10$ V, $\sigma_\text{v} = \frac{5}{\sqrt{2}} = 3.5355$, $b = 4$
$\text{SNR/dB} = 25.8490$ dB (2 P)
Linear value $= 384.503$ (1 P)

(c) Given is the 4-point DFT of a filter impulse response $h(n)$ as

$$H(\mu) = \{4, 2j, 2, -2j\}$$

and an input signal $x(n) = \{3, 1, -1, 5\}$.

(i) Find the 4-point DFT $X(\mu)$ for $x(n)$ for $\mu = 0 \dots 3$. (5 P)

$X(0) = 3 + 1 - 1 + 5 = \mathbf{8}$ (1 P)
$X(1) = 3 - j + 1 + 5j = \mathbf{4{+}4j}$ (1 P)
$X(2) = 3 - 1 - 1 - 5 = \mathbf{-4}$ (1 P)
$X(3) = 3 + j + 1 - 5j = \mathbf{4{-}4j}$ (1 P)
$X(\mu) = \{8, 4 + 4j, -4, 4 - 4j\}$ (1 P)

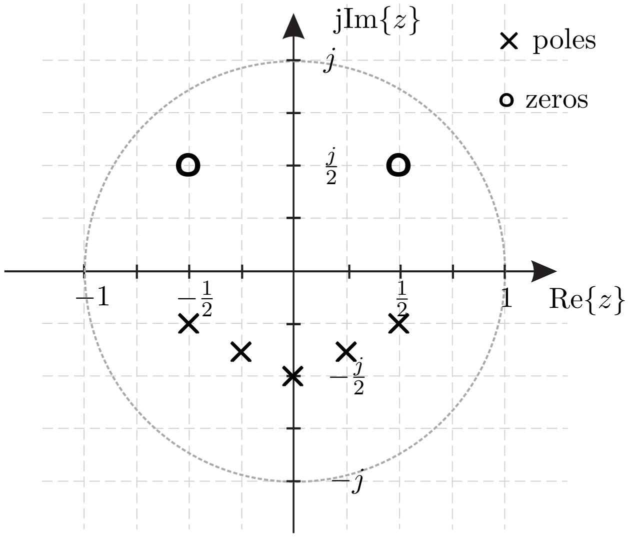
(ii) Find the output signal $y(n) = x(n) \circledast h(n)$ by first finding $Y(\mu) = \text{DFT}\{y(n)\}$. (6 P)

$Y(\mu) = X(\mu) \cdot H(\mu)$
$Y(\mu) = \{8, 4+4j, -4, 4-4j\} \cdot \{4, 2j, 2, -2j\}$
$Y(\mu) = \{32, -8+8j, -8, -8-8j\}$ (2 P)
$y(0) = \frac{1}{4} \cdot (32 + (-8+8j) + (-8) + (-8-8j)) = \mathbf{2}$ (1 P)
$y(1) = \frac{1}{4} \cdot (32 + (-8-8j) + 8 + (-8+8j)) = \mathbf{6}$ (1 P)
$y(2) = \frac{1}{4} \cdot (32 + (8-8j) + (-8) + (8+8j)) = \mathbf{10}$ (1 P)
$y(3) = \frac{1}{4} \cdot (32 + (8+8j) + 8 + (8-8j)) = \mathbf{14}$ (1 P)
$y(n) = \{2, 6, 10, 14\}$ (1 P)

Problem 2 (30 points)

This question consists of two parts a and b. They are **not** related to each other and can be solved independently.

(a) Given is the following pole-zero diagram of a system:



The above system has zeros at $[\pm\frac{1}{2}+\frac{j}{2}]$ and poles at $[\pm\frac{1}{2}-\frac{j}{4}, \pm\frac{1}{4}-\frac{3j}{8}, -\frac{j}{2}]$

(i) Is the above shown system stable? Give reason to your answer. (2 P)

The system is stable, since all poles lie inside the unit circle.

(ii) Insert poles and zeros into the above diagram and name them, such that the resulting system is real-valued. Explain what you did and why. (5 P)

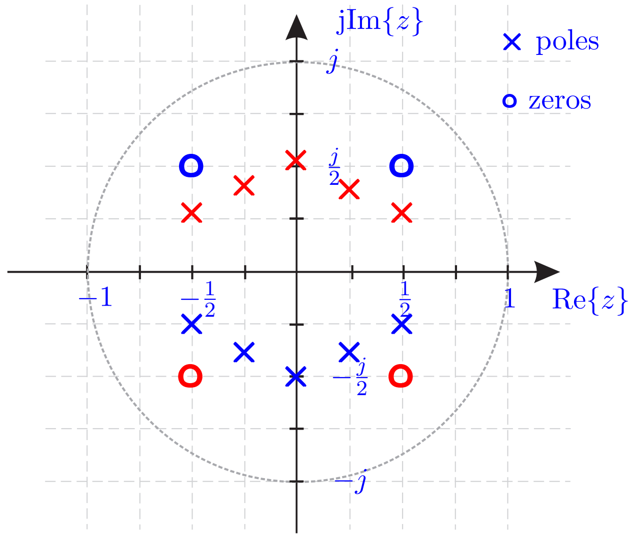
For a real-valued system the poles and zeros have to occur in conjugate complex pairs.
So the poles to add are:

$$z_{\infty,6} = \frac{1}{2} + \frac{j}{4}$$
$$z_{\infty,7} = -\frac{1}{2} + \frac{j}{4}$$
$$z_{\infty,8} = \frac{1}{4} + \frac{3j}{8}$$
$$z_{\infty,9} = -\frac{1}{4} + \frac{3j}{8}$$
$$z_{\infty,10} = \frac{j}{2}$$

And the zeros to add are:

$$z_{0,3} = \frac{1}{2} - \frac{j}{2}$$
$$z_{0,4} = -\frac{1}{2} - \frac{j}{2}$$

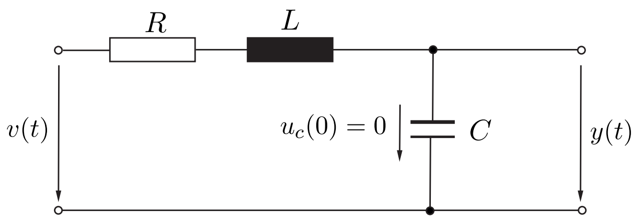
Pole-zero diagram:



(iii) Is the resulting system from (ii) linear phase? Give reason to your answer. (2 P)

The system is not linear phase. For linear phase the zeros occur in pairs mirrored on the unit circle, which is not the case. Additionally the poles cause the system to be not linear phase, they would have to be compensated by poles.

(b) Given is the following network:



It should hold: $L = 2$ H, $R = 6\ \Omega$, $C = 250$ mF

(i) Determine the transfer function $H(s) = \frac{Y(s)}{V(s)}$ for the above Network. (4 P)

Solution using the voltage divider rule:

$$H(s) = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{2}{s^2 + 3s + 2}$$

(ii) The filter $H(s)$ is used as a basis for designing a digital filter $H(z)$ by using the impulse invariance method. Determine the transfer function $H(z)$ for a sampling interval $T = 0.2$. (10 P)

Hint: If you could not solve Problem (i) use $H(s) = \frac{1}{s^2+3s+2}$.

Determination of poles:

$$s_{1/2} = -\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 2} = -\frac{3}{2} \pm \frac{1}{2}$$
$$s_1 = -1$$
$$s_2 = -2$$

Partial fraction expansion:

$$H(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$
$$A_1 = H(s)(s+1)|_{s=-1} = \left.\frac{2}{s+2}\right|_{s=-1} = 2$$
$$A_2 = H(s)(s+2)|_{s=-2} = \left.\frac{2}{s+1}\right|_{s=-2} = -2$$
$$\Rightarrow H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

Inverse Laplace transformation:

$$h(t) = \left(\sum_{i=1}^{2} A_i \cdot \mathrm{e}^{-s_{\infty,i}t}\right) \cdot \delta_{-1}(t) = 2\left(\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right) \cdot \delta_{-1}(t)$$

Sampling:

$$h(n) = h(n \cdot T) = 2\left(\mathrm{e}^{-0.2n} - \mathrm{e}^{-0.4n}\right) \cdot \gamma_{-1}(n)$$

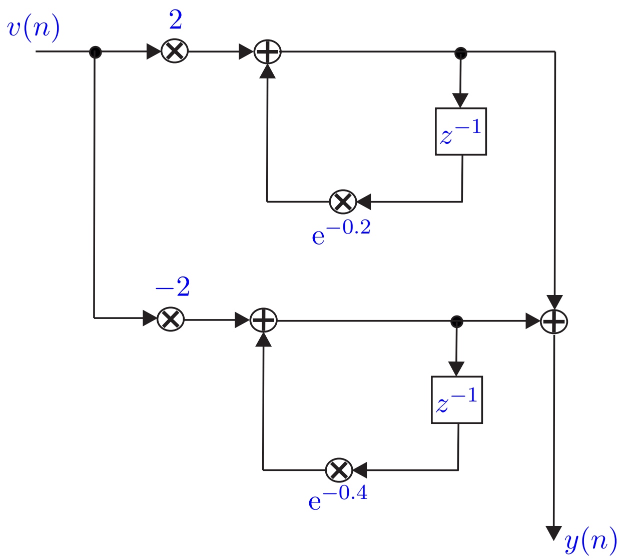
Z-transform:

$$H(z) = \sum_{i=1}^{2} \frac{A_i}{1 - e^{-s_{\infty,i}T z^{-1}}}$$
$$= \frac{2}{1 - e^{-0.2} z^{-1}} - \frac{2}{1 - e^{-0.4} z^{-1}}$$
$$= \frac{2\left(e^{-0.2} - e^{-0.4}\right) z}{z^2 - \left(e^{-0.2} + e^{-0.4}\right) z + e^{-0.6}}$$

(iii) What kind of filter is represented by the system (FIR/IIR)? Is the system a lowpass, bandpass, highpass, bandstop filter? Give reason to both of your answers. (3 P)

The filter represented by this system is an IIR filter (second order), since there are only feedback paths.
The represented filter is a lowpass, since the two poles of the system lie on the positive real part axis and the system has no zeros.

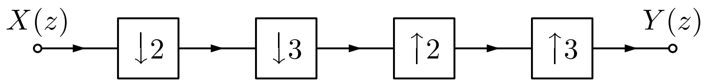
(iv) Sketch the signal flow graph of the parallel structure of the resulting discrete system in direct form I. (4 P)



Problem 3 (40 points)

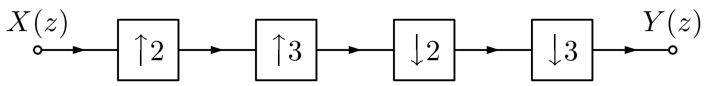
(a) Simplify as much as possible and give the solution in the frequency domain.

(i) System 1: (5 P)



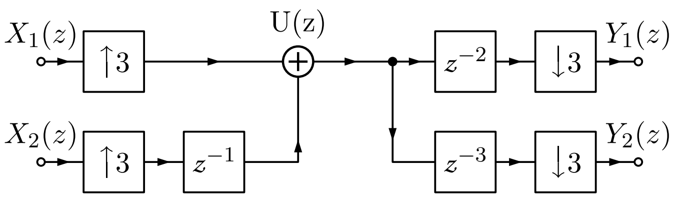
$$Y(z^6) = 1/6 \sum_{k=0}^{5} X(z^{1/6} W_6^k)$$

(ii) Sytem 2: (5 P)



$Y(z) = X(z)$

Given is the following signal flow chart:



(b) How can $y_1(n)$ be described as function of $x_1(n)$ and $x_2(n)$ and how can $y_2(n)$ be described as function of $x_2(n)$ and $x_1(n)$? (18 P)

$$U(z) = X_1(z^3) + X_2(z^3) \cdot z^{-1} \tag{1}$$

$$Y_1(z) = 1/3 \sum_{k=0}^{2} U(z^{1/3} w_3^k) \cdot (z^{1/3} \cdot w_3^k)^{-2} \tag{2}$$

$$= 1/3 \sum_{k=0}^{2} \left(X_1 \left((z^{1/3} w_3^k)^3 \right) + X_2 \left((z^{1/3} w_3^k)^3 \right) \cdot (z^{1/3} w_3^k)^{-1} \right) \cdot (z^{1/3} \cdot w_3^k)^{-2} \tag{3}$$

$$= 1/3 \sum_{k=0}^{2} X_1 \left((z^{1/3} w_3^k)^3 \right) \cdot (z^{1/3} \cdot w_3^k)^{-2} + X_2 \left((z^{1/3} w_3^k)^3 \right) \cdot (z^{1/3} w_3^k)^{-3} \tag{4}$$

$$= 1/3 \sum_{k=0}^{2} X_1 \left((z^{1/3} w_3^k)^3 \right) \cdot (z^{1/3} \cdot w_3^k)^{-2} + X_2(z) \cdot z^{-1} \tag{5}$$

$$= X_2(z) \cdot z^{-1} + 1/3 \cdot X_1(z) \cdot z^{-2/3} \underbrace{(1 + e^{j4\pi/3} + e^{j8\pi/3})}_{=0} \tag{6}$$

$$= X_2(z) \cdot z^{-1} \tag{7}$$

$$\mathcal{Z}^{-1}(X_2(z) \cdot z^{-1}) = x_2[n-1] \tag{8}$$

$$Y_2(z) = 1/3 \sum_{k=0}^{2} U(z^{1/3} w_3^k) \cdot (z^{1/3} \cdot w_3^k)^{-3} \tag{9}$$

$$= 1/3 \sum_{k=0}^{2} \left(X_1 \left((z^{1/3} w_3^k)^3 \right) + X_2 \left((z^{1/3} w_3^k)^3 \right) \cdot (z^{1/3} w_3^k)^{-1} \right) \cdot (z^{1/3} \cdot w_3^k)^{-3} \tag{10}$$

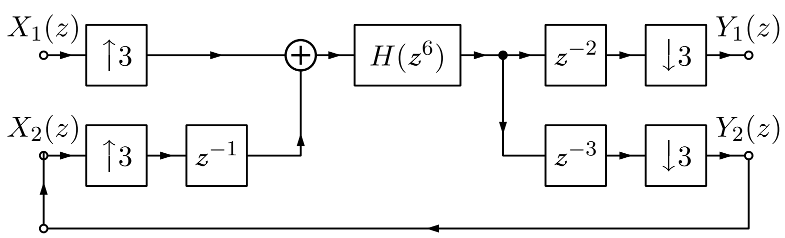
$$= 1/3 \sum_{k=0}^{2} X_1 \left((z^{1/3} w_3^k)^3 \right) \cdot (z^{1/3} \cdot w_3^k)^{-3} + X_2 \left((z^{1/3} w_3^k)^3 \right) \cdot (z^{1/3} w_3^k)^{-4} \tag{11}$$

$$= X_1(z) \cdot z^{-1} + 1/3 \cdot X_2(z) \cdot z^{-4/3} \underbrace{(1 + e^{j8\pi/3} + e^{j16\pi/3})}_{=0} \tag{12}$$

$$= X_1(z) \cdot z^{-1} \tag{13}$$

$$\mathcal{Z}^{-1}(X_1(z) \cdot z^{-1}) = x_1[n-1] \tag{14}$$

Now assume the following signal flow chart



(c) How does $Y_1(z)$ depends of $X_1(z)$? (12 P)

$$
\begin{aligned}
Y_1(z) &= && (15)\\
&= 1/3\sum_{k=0}^{2}\left(X_1\left((z^{1/3}w_3^k)^3\right) + X_2\left((z^{1/3}w_3^k)^3\right)\cdot(z^{1/3}w_3^k)^{-1}\right)\cdot(z^{1/3}\cdot w_3^k)^{-2}\cdot H((z^{1/3}w_3^k)^6) && (16)\\
&= 1/3\sum_{k=0}^{2} X_1\left((z^{1/3}w_3^k)^3\right)\cdot(z^{1/3}\cdot w_3^k)^{-2}\cdot H((z^{1/3}w_3^k)^6) + X_2\left((z^{1/3}w_3^k)^3\right)\cdot(z^{1/3}w_3^k)^{-3}\cdot H(z^2) && (17)\\
&= X_2(z)\cdot z^{-1} + 1/3\cdot X_1(z)\cdot H(z^2)\cdot z^{-2/3}\underbrace{(1+e^{j4\pi/3}+e^{j8\pi/3})}_{=0}\cdot H(z^2) && (18)\\
&= X_2(z)\cdot z^{-1}\cdot H(z^2) && (19)\\
Y_2(z) &= 1/3\sum_{k=0}^{2} U(z^{1/3}w_3^k)\cdot(z^{1/3}\cdot w_3^k)^{-3}\cdot H((z^{1/3}w_3^k)^6) && (20)\\
&= 1/3\sum_{k=0}^{2}\left(X_1\left((z^{1/3}w_3^k)^3\right) + X_2\left((z^{1/3}w_3^k)^3\right)\cdot(z^{1/3}w_3^k)^{-1}\right)\cdot(z^{1/3}\cdot w_3^k)^{-3}\cdot H((z^{1/3}w_3^k)^6) && (21)\\
&= 1/3\sum_{k=0}^{2} X_1\left((z^{1/3}w_3^k)^3\right)\cdot(z^{1/3}\cdot w_3^k)^{-3}\cdot H((z^{1/3}w_3^k)^6) + X_2\left((z^{1/3}w_3^k)^3\right)\cdot(z^{1/3}w_3^k)^{-4}\cdot H(z^2) && (22)\\
&= X_1(z)\cdot z^{-1}\cdot H(z^2) + 1/3\cdot X_2(z)\cdot z^{-4/3}\underbrace{(1+e^{j8\pi/3}+e^{j16\pi/3})}_{=0}\cdot H(z^2) && (23)\\
&= X_1(z)\cdot z^{-1}\cdot H(z^2) && (24)\\
Y_2(z) &= X_2(z) && (25)\\
\Rightarrow Y_1(z) &= X_1(z)\cdot z^{-1}\cdot H(z^2)\cdot z^{-1}\cdot H(z^2) && (26)\\
&= H^2(z^2)\cdot z^{-2}\cdot X_1(z) && (27)
\end{aligned}
$$