Advanced Digital Signal Processing

Solution SS 2015

Examiner: Prof. Dr.-Ing. Gerhard Schmidt

Date: 08.09.2015

Name: ____________________________

Matriculation Number: ____________________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorised to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ____________________________

Marking

Problem	1	2	3
Points	/35	/36	/29

Total number of points: ________________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated ________________ Signature: ____________________________

Solution Problem 1 (35 points)

(1) Figure 1 shows the spectrum of signal $v_1(t)$ (4 P)

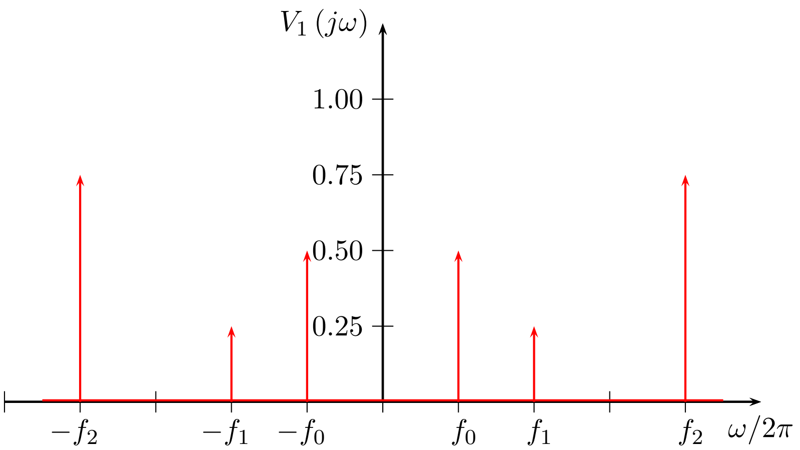

Figure 1: Spectrum $V_1(j\omega)$

(b) Figure 2 shows the spectrum $V_2(j\omega)$ (3 P)

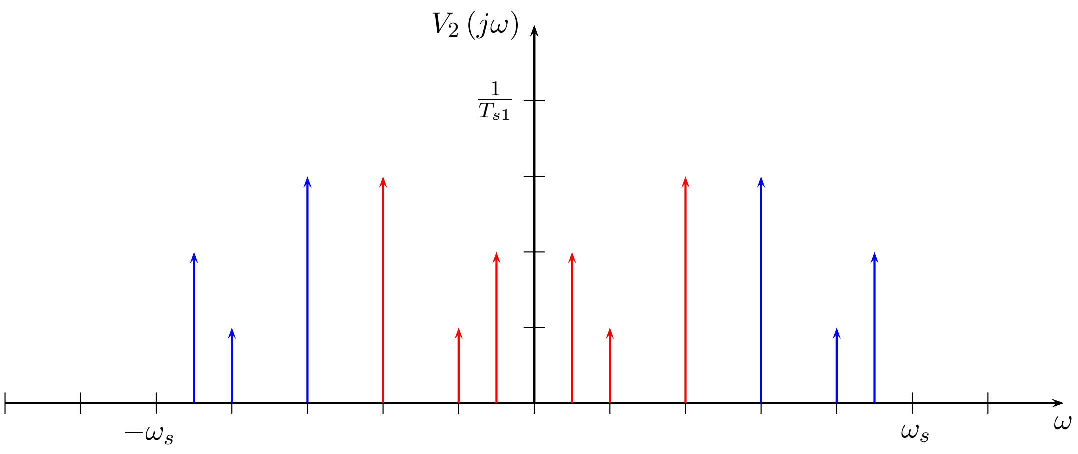

Figure 2: Spectrum $V_2(j\omega)$, $f_{s1} = 10\text{kHz}$. For clearence the replicas are shown in blue

(c) Figure 2 shows the spectrum $V_3(j\omega)$ (3 P)

(d) Only signal $v_{1,s1}(t)$ could be be recovered correctly. (2 P)

(e) Range of the signal $R = \pm 0.5\text{V}$, Number of quantization levels L (3 P)

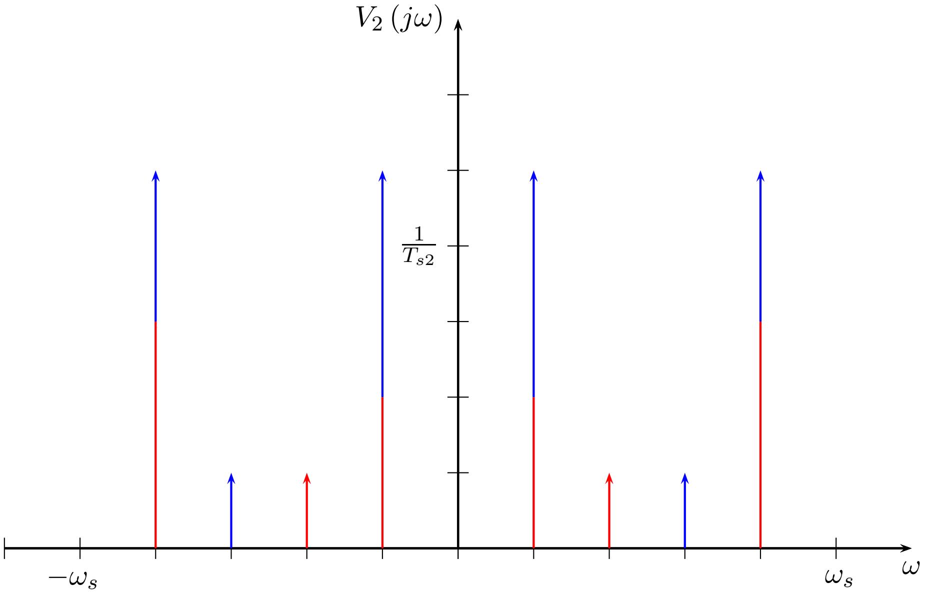

Figure 3: Spectrum $V_3(j\omega)$, $f_{s1} = 5$kHz. The replicas (blue) overlap with the original spectrum (red).

$$L = 2^b = 2^3 = 8$$

Quantization step Δ

$$\Delta = R/L = 1/8 = 0.125\text{V}$$

(f) The midrise quantizer is shown in Figure 4. (4 P)

(g) For $n = 1200$ (2 P)

$$\begin{aligned} \upsilon(n) &= 0.466 \\ \upsilon_q(n) &= Q[\upsilon(n)] \\ &= 0.5 \\ e_q(n) &= \upsilon(n) - \upsilon_q(n) \\ &= 0.034 \end{aligned}$$

(h) The real system is shown in Figure 5 (2 P)

The mathematical model is shown in Figure 6

(i) $X(\mu) = \{+j, 3, -6 + j, 3 + 2j\}$ (5 P)

(j) $Y(\mu) = \{2j, -3j, 0, -2 + 3j\}$ (7 P)
and after further calculations:
$y(n) = \{-0.5 + 0.5j, 1.5 + j, 0.5 + 0.5j, -1.5\}$

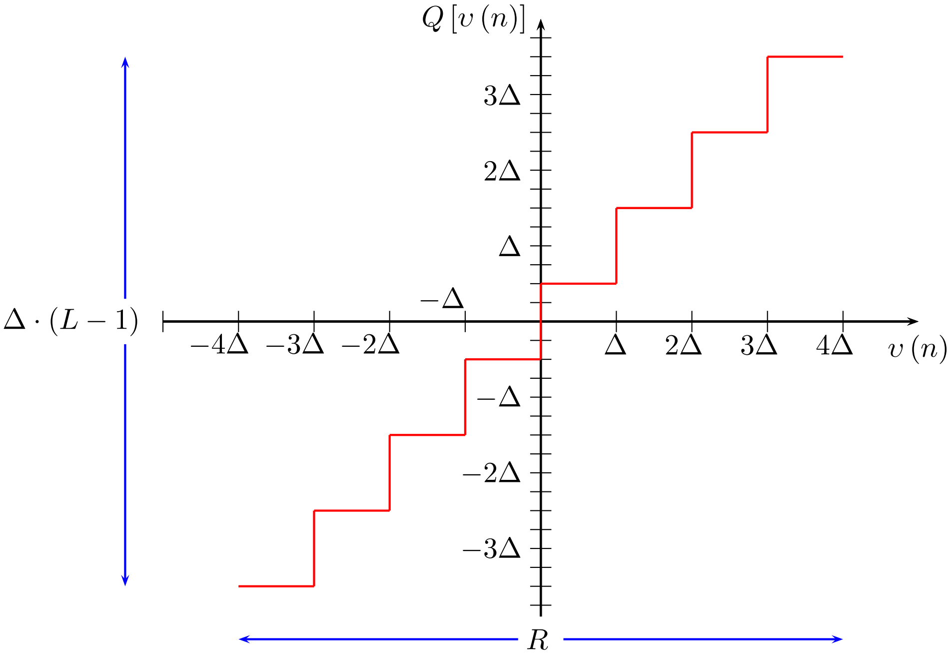

Figure 4: Midrise quantizer with parameters calculated before

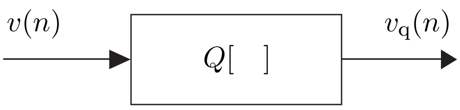

Figure 5: Real system of the quantizer

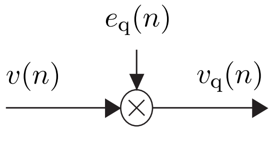

Figure 6: Mathematical model of the quantizer

Solution Problem 2 (36 points)

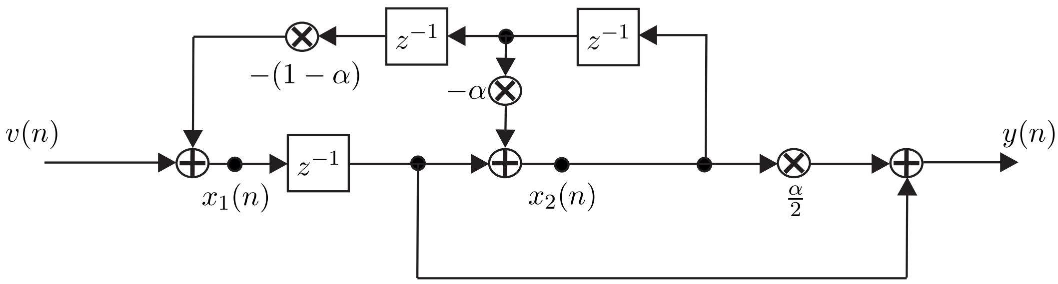

Figure 7: Signal flow graph of the system $H(z)$ with help signals $x_1(n)$ and $x_2(n)$.

(a) For solving the signal flow graph in a first step additional signals $x_1(n)$ and $x_2(n)$ are defined like shown in Figure 7.

For determining the transfer function $H(z)$ all signal will be treated in the z-Domain. The output $Y(z)$ is determined by the following equation:

$$Y(z) = \frac{\alpha}{2} X_2(z) + X_1(z) z^{-1} \tag{1}$$

In a second step the supporting signals $X_1(z)$ and $X_2(z)$ are determined:

$$X_1(z) = -(1-\alpha) X_2(z) z^{-2} + V(z) \tag{2}$$

and

$$\begin{aligned} X_2(z) &= X_1(z) z^{-1} - \alpha X_2(z) z^{-1} \\ &= \frac{1}{1+\alpha z^{-1}} X_1(z) z^{-1} \end{aligned} \tag{3}$$

Including equation 3 in equation 2 and equation 1 leads to:

$$Y(z) = \frac{\frac{\alpha}{2}}{1+\alpha z^{-1}} X_1(z) z^{-1} + X_1(z) z^{-1}$$

$$= X_1(z) \frac{\left(1+\frac{\alpha}{2}\right) z^{-1} + \alpha z^{-2}}{1+\alpha z^{-1}} \tag{4}$$

and

$$X_1(z) = \frac{-(1-\alpha)}{1+\alpha z^{-1}} X_1(z) z^{-3} + V(z)$$

$$\left(1 + \frac{(1-\alpha)z^{-3}}{1+\alpha z^{-1}}\right) X_1(z) = V(z)$$

$$X_1(z) = \frac{1}{1 + \frac{(1-\alpha)z^{-3}}{1+\alpha z^{-1}}} V(z)$$

$$X_1(z) = \frac{1+\alpha z^{-1}}{1+\alpha z^{-1} + (1-\alpha)z^{-3}} V(z) \tag{5}$$

Finally equation 5 can be included into equation 4 to determine the Input/Output relation:

$$Y(z) = \left(\frac{\left(1+\frac{\alpha}{2}\right)z^{-1} + \alpha z^{-2}}{1+\alpha z^{-1}}\right)\left(\frac{1+\alpha z^{-1}}{1+\alpha z^{-1}+(1-\alpha)z^{-3}}\right) V(z)$$

$$= \frac{\left(1+\frac{\alpha}{2}\right)z^{-1} + \alpha z^{-2}}{1+\alpha z^{-1}+(1-\alpha)z^{-3}} V(z) \tag{6}$$

Therefore the transfer function $H(z)$ can be determined:

$$H(z) = \frac{Y(z)}{V(z)}$$

$$= \frac{\left(1+\frac{\alpha}{2}\right)z^{-1} + \alpha z^{-2}}{1+\alpha z^{-1}+(1-\alpha)z^{-3}}$$

(b) The difference equation can be determined by rewriting equation 6 and transforming the result back into the time domain:

$$Y(z) + \alpha Y(z) z^{-1} + (1-\alpha) Y(z) z^{-3} = \left(1+\frac{\alpha}{2}\right) V(z) z^{-1} + \alpha V(z) z^{-2}$$

$$\bullet\!\!-\!\!\circ$$

$$y(n) + \alpha y(n-1) + (1-\alpha) y(n-3) = \left(1+\frac{\alpha}{2}\right) v(n-1) + \alpha v(n-2)$$

(c) The transfer function is already in correct form to determine the Direct Form II realization:

$$H(z) = \frac{\left(1+\frac{\alpha}{2}\right)z^{-1} + \alpha z^{-2}}{1+\alpha z^{-1}+(1-\alpha)z^{-3}}$$

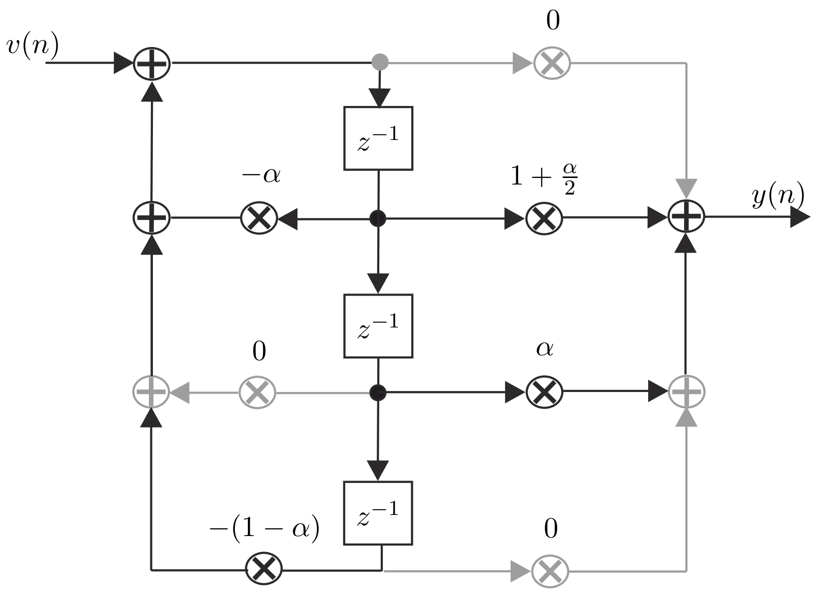

Figure 8: Direct Form II realization of $H(z)$.

(d) The transfer function can be rewritten as:

$$H(z) = \frac{\left(1+\frac{\alpha}{2}\right)z^2 + \alpha z}{z^3 + \alpha z^2 + (1-\alpha)}$$

For stability all Poles have to be inside the unit circle, which means $|z_{\infty,i}| < 1 \quad \forall i$. The poles can be determined by setting the denominator of the transfer function $H(z)$ to zero, which leads to:

$$z^3 + \alpha z^2 + (1-\alpha) = 0$$

Inserting $\alpha = \frac{1}{2}$, leads to:

$$z^3 + \frac{1}{2}z^2 + \frac{1}{2} = 0$$

The first Pole that is given, is:

$$z_{\infty,1} = -1$$

To determine the remaining poles, a polynomal long division is performed:

$$\begin{array}{l}
\left(z^3 + \frac{1}{2}z^2 \qquad + \frac{1}{2}\right) : \left(z+1\right) = z^2 - \frac{1}{2}z + \frac{1}{2} \\
\underline{-z^3 \quad -z^2} \\
\quad -\frac{1}{2}z^2 \\
\quad \underline{\frac{1}{2}z^2 + \frac{1}{2}z} \\
\qquad \frac{1}{2}z + \frac{1}{2} \\
\qquad \underline{-\frac{1}{2}z - \frac{1}{2}} \\
\qquad\qquad 0
\end{array}$$

Afterwards, determining the remaining poles:

$$z^2 - \frac{1}{2}z + \frac{1}{2} = 0$$

This can be solved to:

$$\begin{aligned} z_{\infty,2/3} &= \frac{1}{4} \pm \sqrt{\frac{1}{16} - \frac{1}{2}} \\ &= \frac{1}{4} \pm \sqrt{-\frac{7}{16}} \\ &= \frac{1}{4} \pm \frac{\sqrt{7}}{4}j \end{aligned} \tag{7}$$

This means the filter is marginally stable, because the pole $z_{\infty,1} = -1$ lies on the unit circle and the poles $z_{\infty,2/3} = \frac{1}{4} \pm \frac{\sqrt{7}}{4}j$ lie inside the unit circle.

(e) For stability all Poles have to be inside the unit circle, which means $|z_{\infty,i}| < 1 \quad \forall i$. The poles can be determined by setting the denominator of the transfer function $H(z)$ to zero, which leads to:

$$z^3 - z = 0$$

The first Pole that can directly be seen is:

$$z_{\infty,1} = 0$$

This means, what still has to be solved is:

$$z^2 - 1 = 0$$

This can be solved to:

$$\begin{aligned} z_{\infty,2/3} &= \pm\sqrt{1} \\ &= \pm 1 \end{aligned} \tag{8}$$

This means the filter is marginally stable, because the poles $z_{\infty,2/3} = \pm 1$ lie on the unit circle and the pole $z_{\infty,1} = 0$ lies inside the unit circle.

(f) The bilinear transform is described by the following relationship: $z = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$. For $T = 2$ the relationship is simplified to:

$$z = \frac{1+s}{1-s}. \tag{9}$$

For determining the analogue system $H_\mathrm{a}(s)$ the digital system $H_2(z)$ will be rearranged, additionally the relation $\alpha = 0.5$ is inserted:

$$\begin{aligned} H_1(z) &= \frac{3z^{-1} - z^{-3}}{1 - z^{-2}} \\ &= \frac{3z^2 - 1}{z^3 - z}. \end{aligned} \tag{10}$$

The analogue system is then determined by including equation 9 into equation 10:

$$\begin{aligned}
H_\mathrm{a}(s) &= \frac{3\frac{(1+s)^2}{(1-s)^2} - 1}{\frac{(1+s)^3}{(1-s)^3} - \frac{1+s}{1-s}} \\
&= \frac{3(1+s)^2(1-s) - (1-s)^3}{(1+s)^3 - (1+s)(1-s)^2} \\
&= \frac{3(1+s-s^2-s^3) - (1-3s+3s^2-s^3)}{(1+3s+3s^2+s^3) - (1-s-s^2+s^3)} \\
&= \frac{3+3s-3s^2-3s^3-1+3s-3s^2+s^3}{1+3s+3s^2+s^3-1+s+s^2-s^3} \\
&= \frac{1+3s-3s^2-s^3}{2s+2s^2}.
\end{aligned}$$

(g) odd length $\Rightarrow$ type-1 or type-3 linear-phase filter

(h) $H_2(e^{j\pi}) = 0,\ \sum_k h_2^2(k) \leq 0.2 \Rightarrow$ type-3
$h_2(k) = \{0.1,\ 0.3,\ 0,\ 0,\ 0,\ -0.3,\ -0.1\}$

Solution Problem 3 (29 points)

Hint: The parts (a)-(b), (c)-(d) and (e)-(m) can be solved independently!

Consider the multi-rate system depicted in Figure 9. The spectrum of $X(e^{j\Omega})$ is also depicted in Figure 9.

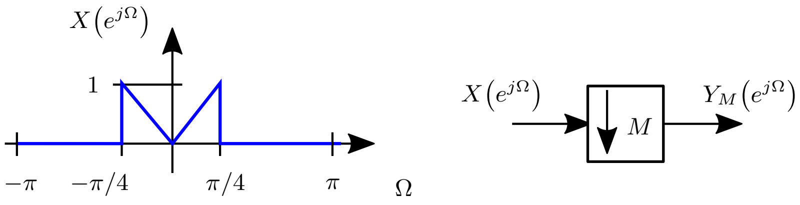

Figure 9: Block diagram of a multi-rate system.

(a) Sketch $Y_M(e^{j\Omega})$ for $M = 2, M = 4$, and $M = 8$ in the range of $-2\pi < \Omega < 2\pi$. For which case(s) does aliasing occur? (6 P)

Aliasing occurs for $M = 8$. The spectra are depicted in Figure 10

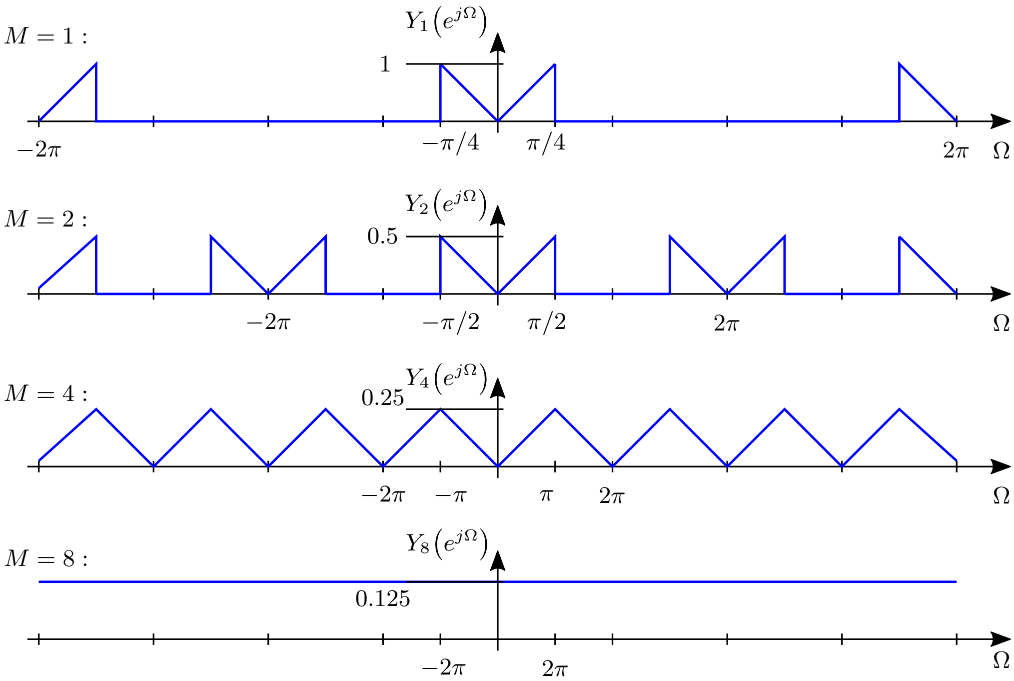

Figure 10: Spectra.

(b) Is it possible to avoid aliasing? Explain **briefly** your answer. (1 P)

Yes, it is possible to avoid aliasing with a help of an anti-alias filter which is an low-pass filter. However, the desired signal components could be attenuated this way since no real filter has an ideal characteristics.

A non-integer downsampling by a factor of $7/3$ can be performed by a multi-rate system.

(c) Explain **briefly** the necessary steps. (2 P)

Non-integer downsampling can be performed by upsampling first and downsample afterwards. For a factor of $7/3$ the input signal would be upsampled first by a factor of 3, filtering with an anti imaging lowpass filter $H(e^{j\Omega})$ with cutoff frequencies $\pm\pi/7$ and then downsampled by a factor of 7. The combined result corresponds to the downsampling by a factor of $7/3$.

(d) Sketch a block diagram. (2 P)

The block diagram is depicted in Figure 11.

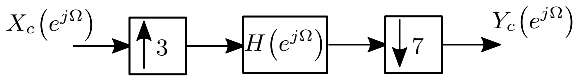

Figure 11: Block diagram of a multi-rate system that performs downsampling by the factor of $7/3$.

Consider the block diagram of a multi-rate system as depicted in Fig. 12.

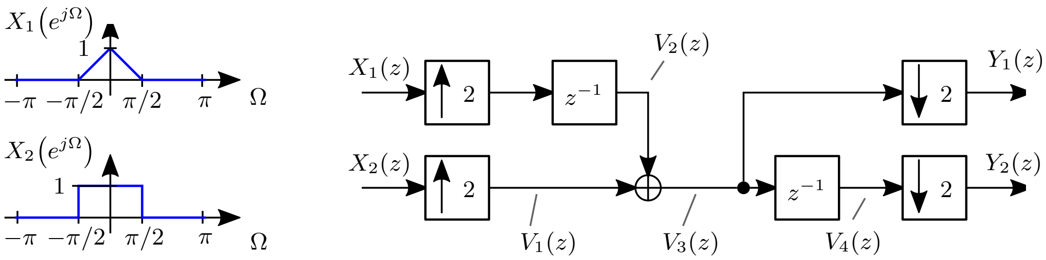

Figure 12: Block diagram of a multi-rate system.

(e) Give an expression for $V_1(z)$ in terms of $X_2(z)$. (1 P)

$$V_1(z) = X_2(z^2) \tag{11}$$

(f) Sketch $\left|V_1(e^{j\Omega})\right|$ in the range of $-2\pi < \Omega < 2\pi$. Label all axes! (2 P)

The spectrum is depicted in Figure 13.

(g) Give an expression for $V_2(z)$ in terms of $X_1(z)$. (1 P)

$$V_2(z) = X_1(z^2)z^{-1} \tag{12}$$

(h) Sketch $\left|V_2(e^{j\Omega})\right|$ in the range of $-2\pi < \Omega < 2\pi$. Label all axes! (2 P)

The spectrum is depicted in Figure 13.

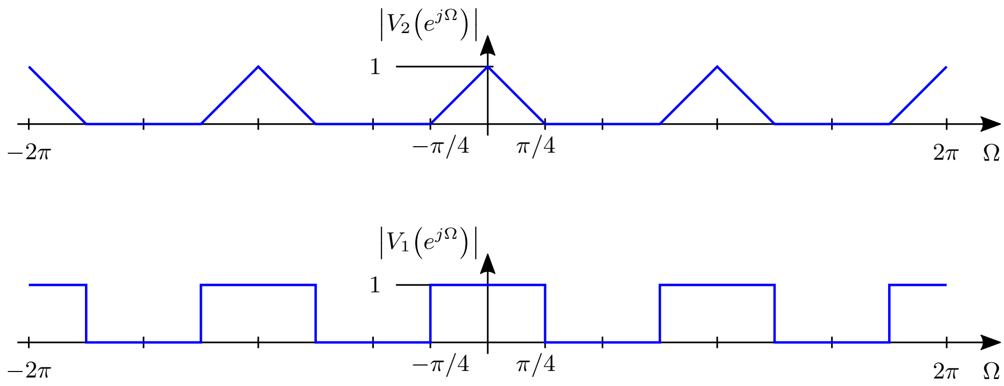

Figure 13: Spectra.

(i) Give an expression for $V_3(z)$ in terms of $X_1(z)$ and $X_2(z)$. (1 P)

$$V_3(z) = X_1(z^2)z^{-1} + X_2(z^2) \tag{13}$$

(j) Give an expression for $V_4(z)$ in terms of $X_1(z)$ and $X_2(z)$. Simplify as much as possible! (2 P)

$$V_4(z) = X_1(z^2)z^{-2} + X_2(z^2)z^{-1} \tag{14}$$

(k) Give an expression for $Y_2(z)$ in terms $X_1(z)$ and $X_2(z)$. Simplify as much as possible! (3 P)

$$Y_2(z) = \frac{1}{2}\left(V_4(z^{\frac{1}{2}}e^{-j\frac{2\pi}{2}0}) + V_4(z^{\frac{1}{2}}e^{-j\frac{2\pi}{2}1})\right) \tag{15}$$

$$= \frac{1}{2}\left(V_4(z^{\frac{1}{2}}) + V_4(-z^{\frac{1}{2}})\right) \tag{16}$$

$$= \frac{1}{2}\left(X_1(z)z^{-1} + X_2(z)z^{-\frac{1}{2}} + X_1(z)z^{-1} - X_2(z)z^{-\frac{1}{2}}\right) \tag{17}$$

$$= X_1(z)z^{-1} \tag{18}$$

(l) Give an expression for $Y_1(z)$ in terms of $X_1(z)$ and $X_2(z)$. Simplify as much as possible! (3 P)

$$
\begin{aligned}
Y_1(z) &= \frac{1}{2}\left(V_3(z^{\frac{1}{2}}e^{-\frac{2\pi}{2}0}) + V_3(z^{\frac{1}{2}}e^{-\frac{2\pi}{2}1})\right) && (19)\\
&= \frac{1}{2}\left(V_3(z^{\frac{1}{2}}) + V_3(-z^{\frac{1}{2}})\right) && (20)\\
&= \frac{1}{2}\left(X_1(z)z^{-\frac{1}{2}} + X_2(z) - X_1(z)z^{-\frac{1}{2}} + X_2(z)\right) && (21)\\
&= X_2(z) && (22)
\end{aligned}
$$

(m) Sketch a simplified block diagram of the system under consideration of the previous results. (3 P)

The block diagram is depicted in Figure 14.

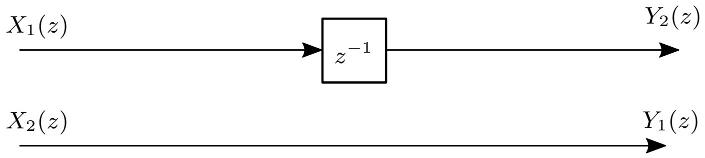

Figure 14: Block diagram of the simplified multi-rate system.